Commutativity of Term Rewriting Systems*

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Abstract

Commutativity is very useful in showing the Church-Rosser property for the union of term rewriting systems. This paper studies the critical pair technique for proving commutativity of term rewriting systems. Extending the concept of critical pairs between two term rewriting systems, a sufficient condition for commutativity is proposed. Using this condition, a new sufficient condition is offered for the Church-Rosser property of left-linear term rewriting systems.

1 Introduction

We consider the commutative property of two term rewriting systems R_1 and R_2 [12]. Hindley [3] and Rosen [12] first studied commutative reduction systems by considering how to infer the Church-Rosser property for a complex system from various properties of its parts. They showed that if R_1 and R_2 commute and have the Church-Rosser property, then the union $R_1 \cup R_2$ also has the Church-Rosser property.

Simple sufficient conditions for commutativity or quasi-commutativity of linear term rewriting systems R_1 and R_2 have been proposed [2, 6, 7, 11, 13]: For example, if two left-linear term rewriting systems R_1 and R_2 do not overlap, then they commute [11, 13]. However, these works were done on the following restrictions: R_1 and R_2 are nonoverlapping with each other [2, 11, 13], or R_1 is $(E-)$ terminating [6, 7]. Hence new conditions are needed to prove commutativity if the systems do not satisfy these restrictions.

This paper studies commutativity of left-linear term rewriting systems R_1 and R_2 without the above restrictions. That is, two systems may overlap and be nonterminating. To treat the overlapping and terminating case, the critical pair concept used to infer the Church-Rosser property [4, 5, 9, 12] is extended. This extension is done by introducing the critical pairs between R_1 and R_2 and classifying them into

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two kinds of pairs; outside pairs and inside pairs. These extended critical pairs are used to propose a sufficient condition for commutativity of term rewriting systems. The proposed result can also be applied to inferring the Church-Rosser property. A new sufficient condition is offered for the Church-Rosser property of left-linear term rewriting systems with overlapping rules.

In Section 2, we present preliminary concepts for term rewriting systems and extend the critical pair concept. Section 3 gives the sufficient conditions for commutativity and for the Church-Rosser property of left-linear term rewriting systems.

2 Term Rewriting Systems

We explain notions of reduction systems and term rewriting systems, and give definitions used in subsequent sections.

2.1 Reduction Systems

A reduction system is a structure $R = \langle A, \rightarrow \rangle$ consisting of some object set A and some binary relation $\rightarrow$ on A (i.e. $\rightarrow \subseteq A \times A$), called a reduction relation. A reduction (starting with x_0) in R is a finite or infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. $\equiv$ denotes the identity of elements of A (or syntactical equality). $\xrightarrow{*}$ is the transitive reflexive closure of $\rightarrow$, $\xrightarrow{\equiv}$ is the reflexive closure of $\rightarrow$, and $=$ is the equivalence relation generated by $\rightarrow$ (i.e. the transitive reflexive symmetric closure of $\rightarrow$). If $x \in A$ is minimal with respect to $\rightarrow$, i.e. $\neg \exists y \in A[x \rightarrow y]$, then x is called a normal form. $NF_{\rightarrow}$ or NF is the set of normal forms. If $x \xrightarrow{*} y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x.

Definition. $R = \langle A, \rightarrow \rangle$ is terminating iff every reduction in R terminates, i.e. there is no infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$.

Definition. $R = \langle A, \rightarrow \rangle$ has the Church-Rosser property (denoted by $CR(R)$) iff $\forall x, y, z \in A[x \xrightarrow{*} y \wedge x \xrightarrow{*} z \Rightarrow \exists w \in A, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.

We express this property with the diagram in Figure 1. In this sort of diagram, dashed arrows denote (existential) reductions depending on the (universal) reductions shown by full arrows.

$$
\begin{tikzcd}
x \arrow[rr, "*"] \arrow[dd, "*"'] & & z \arrow[dd, dashed, "*"] \\
& & \\
y \arrow[rr, dashed, "*"] & & w
\end{tikzcd}
$$

Figure 1

The following properties are well known [1, 8, 4].

Properties. Let R have the Church-Rosser property. Then

(1) the normal form of any element, if it exists, is unique;

(2) $\forall x, y \in A[x = y \Rightarrow \exists w \in A, x \xrightarrow{*} w \wedge y \xrightarrow{*} w]$.

Let $R_1 = \langle A, \underset{1}{\rightarrow} \rangle$ and $R_2 = \langle A, \underset{2}{\rightarrow} \rangle$ be two abstract reduction systems having the same object set A.

Definition. $R_1 = \langle A, \underset{1}{\rightarrow} \rangle$ commutes with $R_2 = \langle A, \underset{2}{\rightarrow} \rangle$ (denoted by $COM(R_1, R_2)$) iff R_1 and R_2 satisfy the diagram in Figure 2.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
x \arrow[r, "*", "2"'] \arrow[d, "*"', "1"] & z \arrow[d, dashed, "1"', "*"] \\
y \arrow[r, dashed, "*", "2"'] & w
\end{tikzcd}$$

Figure 2

Note that R has the Church-Rosser property iff R is self-commuting, i.e. R commutes with itself. Hindley [3] and Rosen [12] discovered the following useful theorem.

Commutative Union Theorem. Let $R_i = \langle A, \underset{i}{\rightarrow} \rangle \quad (i \in I)$ be reduction systems. Let R_i commute with R_j for all $i, j \in I$. Then $\bigcup_{i \in I} R_i$ has the Church-Rosser property, where $\bigcup_{i \in I} R_i = \langle A, \bigcup_{i \in I} \underset{i}{\rightarrow} \rangle$.

Hindley [3] and Rosen [12] also proposed the following sufficient condition for commutativity which enhances the usefulness of the above theorem.

Commutative Lemma. Let R_1 and R_2 satisfy the diagram in Figure 3. Then R_1 commutes with R_2.

$$\begin{tikzcd} x \arrow[r, "2"'] \arrow[d, "1"] & z \arrow[d, dashed, "1"', "*"] \\ y \arrow[r, dashed, "2"', "\equiv"] & w \end{tikzcd}$$

Figure 3

2.2 Term Rewriting Systems

The following explains term rewriting systems that are reduction systems having a term set as an object set A.

Let F be an enumerable set of function symbols denoted by $f, g, h, \cdots$. Let V be an enumerable set of variable symbols denoted by $x, y, z, \cdots$ where $F \cap V = \phi$. $T(F, V)$ denotes the set of terms constructed from F and V. An arity function ρ is a mapping from F to natural numbers $\mathbf{N}$. If $\rho(f) = n$ then f is called an n-ary function symbol. In particular, a 0-ary function symbol is called a constant.

The set $T(F, V)$ of terms on a function symbol set F is inductively defined as follows:

(1) $x \in T(F, V)$ if $x \in V$,

(2) $f \in T(F, V)$ if $f \in F$ and $\rho(f) = 0$,

(3) $f(M_1, \ldots, M_n) \in T(F, V)$ if $f \in F, \rho(f) = n > 0$, and $M_1, \ldots, M_n \in T(F, V)$.

We use T for $T(F, V)$ when F is clear and does not require identification.

A substitution θ is a mapping from a term set T to T such that;

(1) $\theta(f) = f$ if $f \in F$ and $\theta(f) = 0$,

(2) $\theta(f(M_1, \ldots, M_n)) \equiv f(\theta(M_1), \ldots, \theta(M_n))$ if $f(M_1, \ldots, M_n) \in T$.

Thus for term M, $\theta(M)$ is determined by its values on the variable symbols occurring in M. Following common usage, we write this as $M\theta$ rather than $\theta(M)$.

Consider an extra constant $\Box$ called a hole and the set $T(F \cup \{\Box\}, V)$. Then $C \in T(F \cup \{\Box\}, V)$ is called a context on F. We use the notation $C[\ , \ldots, \ ]$ for the context containing n holes ($n \geq 0$). If $N_1, \ldots, N_n \in T(F, V)$, then $C[N_1, \ldots, N_n]$ denotes the result of placing $N_1, \ldots, N_n$ in the holes of $C[\ ,\ldots,\ ]$ from left to right. In particular, $C[\ ]$ denotes a context containing precisely one hole.

N is called a subterm of $M \equiv C[N]$. If N be a subterm occurrence of M, then we write $N \subseteq M$. If $N \not\equiv M$, then we write $N \subset M$. If N_1 and N_2 are subterm occurrences of M having no common symbol occurrences, i.e. $M \equiv C[N_1, N_2]$, then N_1, N_2 are called disjoint, denoted by $N_1 \perp N_2$.

A rewriting rule on T is a pair $\langle M_l, M_r \rangle$ of terms in T such that $M_l \notin V$ and any variable in M_r also occurs in M_l. $\triangleright$ denotes a set of rewriting rules on T, and we write $M_l \triangleright M_r$ for $\langle M_l, M_r \rangle \in \triangleright$. A $\rightarrow$redex, or redex, is a term $M_1\theta$, where $M_l \triangleright M_r$. In this case $M_r\theta$ is called a $\rightarrow$contractum, of $M_l\theta$. The set $\triangleright$ of rewriting rules on T defines a reduction relation $\rightarrow$ on T as follows:

$$M \rightarrow N \text{ iff } M \equiv C[M_l\theta],\ N \equiv C[M_r\theta], \text{ and } M_l \triangleright M_r \\ \text{for some } M_l, M_r, C[\ ], \text{ and } \theta.$$

$M \xrightarrow{A} N$ is written to specify the redex occurrence $A \equiv M_l\theta$ of M in this reduction.

Definition. A term rewriting system R on T is a reduction system $R = \langle T, \rightarrow \rangle$ such that the reduction relation $\rightarrow$ is defined by a set $\triangleright$ of rewriting rules on T. If R has $M_l \triangleright M_r$, then we write $M_l \triangleright M_r \in R$.

For a term rewriting system R, the parallel reduction $\twoheadrightarrow\!\!\!\!\!\!\shortmid\!\shortmid$ for disjoint redex occurrences is defined as follows. Let $M \equiv C[A_1, \cdots, A_m]$ and let $A_i \xrightarrow{A_i} B_i$ $(i = 1, \cdots, m)$. Let $N \equiv C[B_1, \cdots, B_m]$. Then we write $M \twoheadrightarrow\!\!\!\!\!\!\shortmid\!\shortmid N$ or $M \overset{A_1,\cdots,A_m}{\twoheadrightarrow\!\!\!\!\!\!\shortmid\!\shortmid} N$.

If every variable in term M occurs only once, then M is called linear. R is called left-linear iff M_l is linear for any $M_l \triangleright M_r \in R$.

Let $R_1 = \langle T, \underset{1}{\rightarrow} \rangle$ with $\underset{1}{\triangleright}$ and $R_2 = \langle T, \underset{2}{\rightarrow} \rangle$ with $\underset{2}{\triangleright}$ be two term rewriting systems. Then their union can be obtained by $R_1 \cup R_2 = \langle T, \rightarrow \rangle$ with $\underset{1}{\triangleright} \cup \underset{2}{\triangleright}$.

2.3 Critical Pairs

The critical pair concept [4, 5, 12] for a term rewriting system will be extended into a concept for two systems. Let R_1 and R_2 be two term rewriting systems and let $P \triangleright Q \in R_1$ and $M \triangleright N \in R_2$. It may be assumed that the variables have been renamed appropriately, so that P and M share no variables. Assume $S \notin V$ is a subterm occurrence in M, i.e. $M \equiv C[S]$, such that S and P are unifiable, i.e. $S\theta \equiv P\theta$, with a minimal unifier θ [4, 9]. Since $M\theta \equiv C[S]\theta \equiv C\theta[P\theta]$, two reductions starting with $M\theta$, i.e. $M\theta \underset{1}{\rightarrow} C\theta[Q\theta] \equiv C[Q]\theta$ and $M\theta \underset{2}{\rightarrow} N\theta$, can be obtained using $P \triangleright Q \in R_1$ and $M \triangleright N \in R_2$ respectively. Then $P \triangleright Q$ is said to overlap $M \triangleright N$, and the pair of terms $\langle C[Q]\theta, N\theta \rangle$ is a critical pair of $P \triangleright Q$ on $M \triangleright N$. The pair is inside (resp. outside) critical if $S \subset M$ (resp. $S \equiv M$). $P \triangleright Q \in R_1$ and $M \triangleright N \in R_2$ may be chosen to be the same rule, but in this case we shall not consider the case $S \equiv M$, which gives the trivial pair $\langle N, N \rangle$. Note that two rules play asymmetrical role in this definition.

$crit(R_1, R_2)$ denotes the set of the critical pairs for all $P \triangleright Q \in R_1$ and $M \triangleright N \in R_2$ such that $P \triangleright Q$ overlaps $M \triangleright N$. $crit_{in}(R_1, R_2)$ and $crit_{out}(R_1, R_2)$ denote the set of inside critical pairs and the set of outside critical pairs respectively. Thus $crit(R_1, R_2) = crit_{in}(R_1, R_2) \cup crit_{out}(R_1, R_2)$. Note that generally $crit(R_1, R_2) \neq crit(R_2, R_1)$ since the definition of overlapping is asymmetrical.

$crit(R)$, $crit_{in}(R)$ and $crit_{out}(R)$ indicate $crit(R, R)$, $crit_{in}(R, R)$ and $crit_{out}(R, R)$ respectively. Thus $crit(R)$ coincides with the set of critical pairs of R defined in [4, 5, 9].

We say that R_1 and R_2 are overlapping with each other if $crit(R_1, R_2) \cup crit(R_2, R_1) \neq \phi$; R_1 and R_2 are nonoverlapping with each other if they are not overlapping with each other. R is overlapping if $crit(R) \neq \phi$; R is nonoverlapping if it is not overlapping. [4, 5, 9].

Remark. Jouannaud and Kirchner [6] and Jouannaud and Munoz [7] also proposed the idea of critical pairs between two systems R_1 and R_2 independently of the author. However, they applied it in a different situation, to discuss the sufficient conditions for the Church-Rosser property and for the termination property of $R_1 \cup R_2$ under the stronger assumptions that R_1 is E-terminating and R_2 is an equational system E. This paper does not assume the termination property of term rewriting systems.

The following sufficient conditions for the Church-Rosser property are well known [4, 5, 9, 12].

Proposition 2.1 (Knuth-Bendix's Condition). Let R be terminating, and let P and Q have the same normal form for any critical pair $\langle P, Q \rangle$ in R. Then R has the Church-Rosser property.

Proposition 2.2 (Rosen's Condition). Let R be left-linear and nonoverlapping. Then R has the Church-Rosser property.

Rosen's condition is a particular case of Huet's condition:

Proposition 2.3 (Huet's Condition). Let R be left-linear. If $P \twoheadrightarrow\!\!\!\!\!\!\;\shortmid\!\shortmid\;\, Q$ for every critical pair $\langle P, Q \rangle$ in R, then R has the Church-Rosser property.

For more discussion concerning the Church-Rosser property of term rewriting systems, see [4, 6, 10, 15].

3 Sufficient Condition for Commutativity

This section shows a sufficient condition for commutativity of two left-linear term rewriting systems R_1 and R_2 on $T(F, V)$. From here on, $\underset{i}{\rightarrow}$ and $\underset{i}{-\!\!\shortparallel\!\!\rightarrow}$ denote the reduction relation and the parallel reduction relation of R_i $(i = 1, 2)$ respectively.

Lemma 3.1. If we have the diagram in Figure 4 then R_1 commutes with R_2.

$$\begin{tikzcd}[column sep=huge, row sep=huge] M \arrow[r, "\shortparallel"', "2"'] \arrow[d, "\shortparallel\ 1"] & P \arrow[d, dashed, "1"', "*"] \\ N \arrow[r, dashed, "\shortparallel"', "2"'] & Q \end{tikzcd}$$

Figure 4

Proof. From $\overset{*}{\underset{1}{-\!\!\shortparallel\!\!\rightarrow}} = \overset{*}{\underset{1}{\rightarrow}}$, we obtain

$\forall M, N, P[M \underset{1}{-\!\!\shortparallel\!\!\rightarrow} N \wedge M \underset{2}{-\!\!\shortparallel\!\!\rightarrow} P \Rightarrow \exists Q, N \underset{2}{-\!\!\shortparallel\!\!\rightarrow} Q \wedge P \overset{*}{\underset{1}{-\!\!\shortparallel\!\!\rightarrow}} Q]$.

By applying the Commutativity Lemma, we can prove commutativity of $\overset{*}{\underset{1}{-\!\!\shortparallel\!\!\rightarrow}}$ and $\overset{*}{\underset{2}{-\!\!\shortparallel\!\!\rightarrow}}$. Since $\overset{*}{\underset{i}{-\!\!\shortparallel\!\!\rightarrow}} = \overset{*}{\underset{i}{\rightarrow}}$ $(i = 1, 2)$, it follows that R_1 commutes with R_2. $\square$

Let $A \equiv C[x_1, \cdots, x_n]$ where no variable occurs in C. Then we say the subterm occurrence P of $A\theta \equiv C[x_1\theta, \cdots, x_n\theta]$ occurs in the substitution θ if P occurs in some $x_i\theta$.

Lemma 3.2. Let $M \equiv A\theta \xrightarrow[1]{M} N \equiv B\theta$, $A \triangleright B \in R_1$, and $M \equiv A\theta \overset{P_1, \cdots, P_p}{\underset{2}{-\!\!\shortparallel\!\!\rightarrow}} P$ where P_i $(i = 1, \cdots, p)$ occurs in θ. Then a term Q can be obtained such that $N \underset{2}{-\!\!\shortparallel\!\!\rightarrow} Q$ and $P \xrightarrow[1]{} Q$ (Figure 5).

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
M \arrow[r, "P_1,\cdots,P_p", "2"', "\shortparallel" marking] \arrow[d, "M"', "1"] & P \arrow[d, dashed, "1"'] \\
N \arrow[r, dashed, "2"', "\shortparallel" marking] & Q
\end{tikzcd}
$$

Figure 5

Proof. Since P_i $(i = 1, \cdots, p)$ occurs in θ, $P \equiv A\theta'$ can be denoted for some θ' such that $x\theta \mathrel{\underset{2}{-\!\!\shortparallel\!\!\!\to}} x\theta'$ for any x in A. Take $Q \equiv B\theta'$. Then it follows that $N \equiv B\theta \mathrel{\underset{2}{-\!\!\shortparallel\!\!\!\to}} Q \equiv B\theta'$ and $P \equiv A\theta' \mathrel{\underset{1}{\longrightarrow}} Q \equiv B\theta'$. $\Box$

Theorem 3.1. Let R_1 and R_2 be left-linear term rewriting systems. Then R_1 commutes with R_2 if R_1 and R_2 satisfy the following conditions:

(1) $\forall \langle P, Q\rangle \in \mathit{crit}(R_1, R_2)\ \exists S\ [P \mathrel{\underset{2}{-\!\!\shortparallel\!\!\!\to}} S \wedge Q \mathrel{\underset{1}{\overset{*}{\to}}} S]$,

(2) $\forall \langle Q, P\rangle \in \mathit{crit}_{in}(R_2, R_1)\ \ [Q \mathrel{\underset{1}{-\!\!\shortparallel\!\!\!\to}} P]$.

Proof. Let $M \mathrel{\overset{A_1,\cdots,A_m}{\underset{1}{-\!\!\shortparallel\!\!\!\to}}} N$ and $M \mathrel{\overset{B_1,\cdots,B_n}{\underset{2}{-\!\!\shortparallel\!\!\!\to}}} P$. If we have the diagram in Figure 6, then the theorem follows from Lemma 3.1. Hence we will show the existence of the term Q in Figure 6 under the above conditions.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
M \arrow[r, "{B_1, \cdots, B_n}", "\|"' marking, "2"'] \arrow[d, "{A_1, \cdots, A_m}"', "=" marking, "1"] & P \arrow[d, dashed, "1"', "*"] \\
N \arrow[r, dashed, "\|" marking, "2"'] & Q
\end{tikzcd}$$

Figure 6

Let $\Gamma = \{A_i | \exists B_j, A_i \subseteq B_j\} \cup \{B_i | \exists A_j, B_i \subseteq A_j\}$ and $\Delta = \{A_i | \forall B_j, A_i \not\subseteq B_j\} \cup \{B_i | \forall A_j, B_i \not\subseteq A_j\}$. Then the redex occurrences $A_1, \cdots, A_m$ and $B_1, \cdots, B_n$ of M are classified into two sets Γ and Δ. The length $|M|$ of a term M is defined by the number of symbols in M. $|\Gamma|$ denotes $\sum_{M \in \Gamma} |M|$. By using induction on $|\Gamma|$, we will prove the existence of Q in Figure 6.

The case $|\Gamma| = 0$ is trivial since $A_1, \cdots, A_m$ and $B_1, \cdots, B_n$ are disjoint. Assume the theorem for $|\Gamma| < k$ $(k > 0)$. We consider the case $|\Gamma| = k$. Let $\Delta = \{M_1, \cdots, M_p\}$. Then we can write $M \equiv C[M_1, \cdots, M_p]$, $N \equiv C[N_1, \cdots, N_p]$, $P \equiv C[P_1, \cdots, P_p]$ where $M_i \mathrel{\underset{1}{-\!\!\!\|\!\!\!\to}} N_i$ and $M_i \mathrel{\underset{2}{-\!\!\!\|\!\!\!\to}} P_i$ $(i = 1, \cdots, p)$. We will now show that for every M_i, we can obtain Q_i satisfying the diagram in Figure 7.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
M_i \arrow[r, "\|" marking, "2"'] \arrow[d, "=" marking, "1"] & P_i \arrow[d, dashed, "1"', "*"] \\
N_i \arrow[r, dashed, "\|" marking, "2"'] & Q_i
\end{tikzcd}$$

Figure 7

There are two cases.

Case 1. $M_i \notin \{B_1, \cdots, B_n\}$.

Then $M_i \xrightarrow[1]{M_i} N_i$ and $M_i \overset{B'_1,\cdots,B'_q}{\underset{2}{-\!\!\|\!\!\!\rightarrow}} P_i$, where $B'_j \in \{B_1, \cdots, B_n\}$ and $B'_j \subset M_i$ for all B'_j. Let $A \triangleright B \in R_1$, $M_i \equiv A\theta$, and $N_i \equiv B\theta$. If every redex occurrence B'_j of M_i occurs in θ then we can obtain Q_i by Lemma 3.2.

Now assume that some B'_j exists which does not occur in θ. Without loss of generality, it may be assumed that B'_1 does not occur in θ. Then there exists $A' \triangleright B' \in R_2$ such that $B'_1 \equiv A'\theta'$. Since $A' \triangleright B'$ overlaps $A \triangleright B$ and $B'_1 \subset M_i$, there is an inside critical pair, say $\langle D, E\rangle$, in $crit_{in}(R_2, R_1)$. Let $M_i \xrightarrow[2]{B'_1} \tilde{M}_i$. Then $\tilde{M}_i \equiv D\theta''$ and $N_i \equiv E\theta''$ for some θ''. From condition (2) of the theorem, $D \underset{1}{-\!\!\|\!\!\!\rightarrow} E$. Hence we have $\tilde{M}_i \overset{C_1,\cdots,C_r}{\underset{1}{-\!\!\|\!\!\!\rightarrow}} N_i$. Also, $\tilde{M}_i \overset{B'_2,\cdots,B'_q}{\underset{1}{-\!\!\|\!\!\!\rightarrow}} P_i$. For the redex occurrences $C_1, \cdots, C_r$ and $B'_2, \cdots, B'_q$ of $\tilde{M}_i$, we take $\Gamma' = \{C_i | \exists B'_j, C_i \subseteq B'_j\} \cup \{B'_i | \exists C_j, B'_i \subseteq C_j\}$. Since $\forall \tilde{B} \in \Gamma'\ \exists B'_j (2 \leq j \leq q)$, $\tilde{B} \subseteq B'_j$, we can easily show that $|\Gamma'| \leq \sum_{j=2}^{q} |B'_j|$. Thus $|\Gamma'| \leq \sum_{j=2}^{q} |B'_j| < \sum_{j=1}^{q} |B'_j| \leq |\Gamma|$. Using the induction hypothesis, we obtain the diagram in Figure 8.

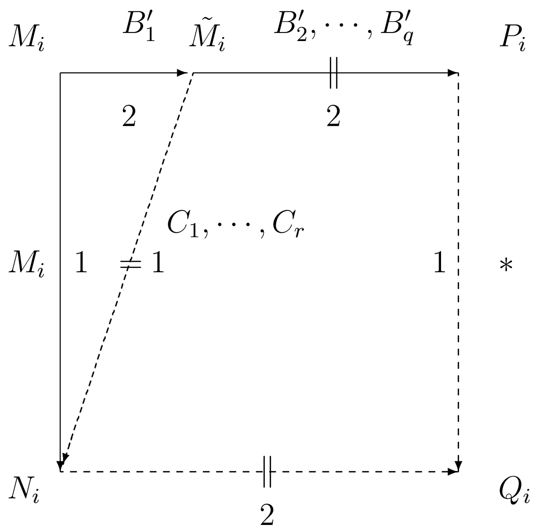

Figure 8

Case 2. $M_i \in \{B_1, \cdots, B_n\}$.

Then $M_i \overset{A'_1,\cdots,A'_q}{\underset{1}{-\!\!\|\!\!\!\rightarrow}} N_i$ and $M_i \xrightarrow[2]{M_i} P_i$, where $A'_j \in \{A_1, \cdots, A_m\}$ and $A'_j \subseteq M_i$ for all A'_j. Let $A \triangleright B \in R_2$, $M_i \equiv A\theta$, and $P_i \equiv B\theta$. If every redex occurrence A'_j of M_i occurs in θ then we can obtain Q_i by Lemma 3.2.

It may be assumed that A'_1 does not occur in θ for the same reason as in case (1). Then there exists $A' \triangleright B' \in R_1$ such that $A'_1 \equiv A'\theta'$. Since $A' \triangleright B'$ overlaps $A \triangleright B$ and $A'_1 \subseteq M_i$, we can obtain a critical pair, say $\langle D, E\rangle$, in $crit(R_1, R_2)$ from this overlapping. Let $M_i \xrightarrow[1]{A'_1} \tilde{M}_i$. Then $P_i \equiv E\theta''$ and $\tilde{M}_i \equiv D\theta''$ for some θ''. From condition (1) of the theorem, there is some S such that $D \underset{2}{\twoheadrightarrow} S$ and $D \xrightarrow[1]{*} S$. Take $\tilde{P}_i \equiv S\theta''$. Then we have $\tilde{M}_i \overset{C_1,\cdots,C_r}{\underset{2}{\twoheadrightarrow}} \tilde{P}_i$ and $P_i \xrightarrow[1]{*} \tilde{P}_i$. Also, $\tilde{M}_i \overset{A'_2,\cdots,A'_q}{\underset{1}{\twoheadrightarrow}} N_i$. For the redex occurrences $A'_2, \cdots, A'_q$ and $C_1, \cdots, C_r$ of $\tilde{M}_i$, we take Γ' in the same way as in case (1); it can be proven that $|\Gamma'| < |\Gamma|$. Using the induction hypothesis, we obtain the diagram in Figure 9.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
M_i \arrow[rr, "M_i", "2"'] \arrow[d, "A'_1"', "1"] & & P_i \arrow[d, dashed, "1"', "*"] \\
\tilde{M}_i \arrow[rr, dashed, "C_1, \cdots, C_r", "2"'] \arrow[d, "A'_2, \cdots, A'_q"', "1"] & & \tilde{P}_i \arrow[d, dashed, "1"', "*"] \\
N_i \arrow[rr, dashed, "2"'] & & Q_i
\end{tikzcd}
$$

Figure 9

Take $Q \equiv C[Q_1, \cdots, Q_p]$. Then it follows that $N \underset{2}{\twoheadrightarrow} Q$ and $P \xrightarrow[1]{*} Q$. $\square$

The following corollary is given in [11, 13].

Corollary 3.1. Let left-linear term rewriting systems R_1 and R_2 be nonoverlapping with each other. Then R_1 commutes with R_2.

Proof. It is obvious from Theorem 3.1. $\square$

Example 3.1. Consider the left-linear term rewriting systems R_1 and R_2:

$$R_1 \quad \begin{cases} f(x) \triangleright h(f(x)) \\ g(x) \triangleright h(g(x)) \end{cases}$$

$$R_2 \quad \begin{cases} f(x) \triangleright g(x) \\ h(f(x)) \triangleright h(g(x)) \end{cases}$$

Then $crit(R_1, R_2) = \{\langle h(f(x)), g(x)\rangle, \langle h(h(f(x))), h(g(x))\rangle\}$ and $crit_{in}(R_2, R_1) = \phi$. It can be shown that $h(f(x)) \underset{2}{\rightarrow} h(g(x))$ and $g(x) \underset{1}{\rightarrow} h(g(x))$ for the critical pair $\langle h(f(x)), g(x)\rangle$, and that $h(h(f(x))) \underset{2}{\rightarrow} h(h(g(x)))$ and $h(g(x)) \underset{1}{\rightarrow} h(h(g(x)))$ for the critical pair $\langle h(h(f(x))), h(g(x))\rangle$. By applying Theorem 3.1, it follows that $COM(R_1, R_2)$.

Let $R = R_1 \cup R_2$. It can easily be shown that $CR(R_1)$ by Proposition 2.2 (Rosen's condition) and $CR(R_2)$ by Proposition 2.1 (Knuth-Bendix's condition). Thus $CR(R)$ can be obtained from the Commutative Union Theorem. Note that non of Propositions 2.1, 2.2, or 2.3 can be directly applied to R. □

Example 3.2. Consider the left-linear term rewriting systems R_1 and R_2:

$$R_1 \quad \begin{cases} f(x) \triangleright g(f(x)) \\ h(x) \triangleright p(h(x)) \end{cases}$$

$$R_2 \quad \begin{cases} f(x) \triangleright h(f(x)) \\ g(x) \triangleright p(p(h(x))) \end{cases}$$

Then $crit(R_1, R_2) = \{\langle g(f(x)), h(f(x))\rangle\}$ and $crit_{in}(R_2, R_1) = \phi$. It can be shown that $g(f(x)) \underset{2}{\rightarrow} p(p(h(f(x))))$ and $h(f(x)) \underset{1}{\rightarrow} p(h(f(x))) \underset{1}{\rightarrow} p(p(h(f(x))))$ for the critical pair $\langle g(f(x)), h(f(x))\rangle$; by applying Theorem 3.1, it follows that R_1 commutes with R_2.

Let $R = R_1 \cup R_2$. We can easily show $CR(R_1)$ and $CR(R_2)$ by Proposition 2.2 (Rosen's condition). Thus $CR(R)$ can be obtained from the Commutative Union Theorem. It is obvious that non of Propositions 2.1, 2.2, or 2.3 can be directly applied to R. □

Since self-commuting $COM(R, R)$ and the Church-Rosser property $CR(R)$ are equivalent, we can obtain the following sufficient condition for the Church-Rosser property from Theorem 3.1.

Corollary 3.2. Let R be a left-linear term rewriting system. Then R has the Church-Rosser property if:

(1) $\forall\langle P, Q\rangle \in crit_{out}(R)\ \exists S\ [P \twoheadrightarrow\!\!\!\!\!+\ S \wedge Q \xrightarrow{*} S]$,

(2) $\forall\langle Q, P\rangle \in crit_{in}(R)\ \ [Q \twoheadrightarrow\!\!\!\!\!+\ P]$.

Proof. Take $R = R_1 = R_2$. Since $crit(R) = crit_{out}(R) \cup crit_{in}(R)$, we can replace condition (1) of Theorem 3.1 with condition (1) of the corollary. Hence the corollary holds. □

Note that Proposition 2.3 (Huet's condition) gives a particular case of Corollary 3.2.

Example 3.3. Consider the left-linear term rewriting system R:

$$R \quad \begin{cases} p(x) \triangleright q(x) \\ p(x) \triangleright r(x) \\ q(x) \triangleright s(p(x)) \\ r(x) \triangleright s(p(x)) \\ s(x) \triangleright f(p(x)) \end{cases}$$

Then $crit_{out}(R) = \{\langle q(x), r(x)\rangle, \langle r(x), q(x)\rangle\}$ and $crit_{in}(R) = \phi$. Since $q(x) \to s(p(x))$ and $r(x) \to s(p(x))$, we can apply Corollary 3.2. Thus it is obtained that R has the Church-Rosser property. Note that the Church-Rosser property of R cannot be proven by applying Proposition 2.1, 2.2, or 2.3. □

4. Conclusion

In this paper we have proposed a new sufficent condition to prove commutativity of left-linear term rewriting systems, by extending the critical pair concept to overlapping rewriting rules. It has been shown that this condition can be applied to proving the Church-Rosser property of left-linear term rewriting systems to which the sufficient conditions proposed by Knuth and Bendix [9], Rosen [12], and Huet [4] cannot directly apply. The proposed result offers a useful means to analyze a complex term rewriting system as the union of simpler systems.

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