On the Church-Rosser Property for the Direct Sum of Term Rewriting Systems

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Abstract

The direct sum of two term rewriting systems is the union of systems having disjoint sets of function symbols. It is shown that if two term rewriting systems both have the Church-Rosser property respectively then the direct sum of these systems also has this property.

1 Introduction

We consider properties of the direct sum system $R_1 \oplus R_2$ obtained from two term rewriting systems R_1 and R_2 [3]. The first study on the direct sum system was conducted by Klop in [3] in order to consider the Church-Rosser property for combinatory reduction systems having nonlinear rewriting rules, which contain term rewriting systems as a special case. He showed that if R_1 is a regular, i.e., linear and nonambiguous, system and R_2 consists of the single nonlinear rule $D(x, x) \triangleright x$, then the direct sum $R_1 \oplus R_2$ has the Church-Rosser property. He also showed in the same manner that if R_2 consists of the nonlinear rules

$$R_2 \quad \begin{cases} \mathrm{if}(T, x, y) \triangleright x \\ \mathrm{if}(F, x, y) \triangleright y \\ \mathrm{if}(z, x, x) \triangleright x \end{cases}$$

then the direct sum $R_1 \oplus R_2$ also has the Church-Rosser property. This result gave a positive answer for an open problem suggested by O'Donnell [4].

Klop's work was done on combinatory reduction systems having the following restrictions: R_1 is a regular (i.e., linear and nonambiguous) system, and R_2 is a nonlinear system having specific rules such as $D(x,x) \triangleright x$. In particular, the restriction on R_1 plays an essential role in his proof of the Church-Rosser property of $R_1 \oplus R_2$; hence his result cannot be applied to combinatory reduction systems (and term rewriting systems) without this restriction.

From Klop's work, we consider the conjecture that these restrictions can be completely removed from R_1 and R_2 in the framework of term rewriting systems [2], i.e., the direct sum of term rewriting systems R_1 and R_2, independent of their properties such as linearity or ambiguity, always preserves their Church-Rosser property. In this paper we shall prove this conjecture: For any two term rewriting systems R_1 and R_2, R_1 and R_2 have the Church-Rosser property iff $R_1 \oplus R_2$ has this property.

2 Notations and Definitions

We explain notions of reduction systems and term rewriting systems, and give definitions for the following sections. We start from abstract reduction systems.

2.1 Reduction Systems

A reduction system is a structure $R = \langle A, \rightarrow \rangle$ consisting of some object set A and some binary relation $\rightarrow$ on A (i.e., $\rightarrow \subset A \times A$), called a reduction relation. A reduction (starting with x_0) in R is a finite or infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. The identity of elements of A (or syntactical equality) is denoted by $\equiv$. $\xrightarrow{*}$ is the transitive reflexive closure of $\rightarrow$, $\xrightarrow{\equiv}$ is the reflexive closure of $\rightarrow$, and $=$ is the equivalence relation generated by $\rightarrow$ (i.e., the transitive reflexive symmetric closure of $\rightarrow$). If $x \in A$ is minimal with respect to $\rightarrow$, i.e., $\neg \exists y \in A[x \rightarrow y]$, then we say that x is a normal form, and let $NF_{\rightarrow}$ or NF be the set of normal forms. If $x \xrightarrow{*} y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x.

Definition. $R = \langle A, \rightarrow \rangle$ is strongly normalizing (denoted by $SN(R)$ or $SN(\rightarrow)$) iff every reduction in R terminates, i.e., there is no infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$.

Definition. $R = \langle A, \rightarrow \rangle$ has the Church-Rosser property (denoted by $CR(R)$) iff $\forall x, y, z \in A[x \xrightarrow{*} y \wedge x \xrightarrow{*} z \Rightarrow \exists w \in A, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.

We express this property with the diagram in Figure 1. In this sort of diagram, dashed arrows denote (existential) reductions depending on the (universal) reductions shown by full arrows.

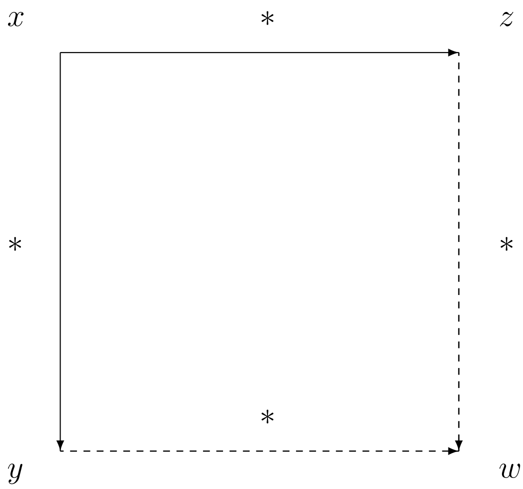

Figure 1

The following properties are well known [1],[2].

Properties. Let R have the Church-Rosser property, then,

(1) the normal form of any element, if it exists, is unique,

(2) $\forall x, y \in A[x = y \Rightarrow \exists w \in A, x \xrightarrow{*} w \wedge y \xrightarrow{*} w]$.

2.2 Term Rewriting Systems

Next, we will explain term rewriting systems that are reduction systems having a term set as an object set A.

Let F be an enumerable set of function symbols denoted by $f, g, h, \cdots$, and let V be an enumerable set of variable symbols denoted by $x, y, z, \cdots$ where $F \cap V = \phi$. By $T(F, V)$, we denote the set of terms constructed from F and V. An arity function ρ is a mapping from F to natural numbers $\mathbf{N}$, and if $\rho(f) = n$ then f is called an n-ary function symbol. In particular, a 0-ary function symbol is called a constant.

The set $T(F, V)$ of terms on a function symbol set F is inductively defined as follows:

(1) $x \in T(F, V)$ if $x \in V$,

(2) $f \in T(F, V)$ if $f \in F$ and $\rho(f) = 0$,

(3) $f(M_1, \ldots, M_n) \in T(F, V)$ if $f \in F, \rho(f) = n > 0$, and $M_1, \ldots, M_n \in T(F, V)$.

We use T for $T(F, V)$ when F is clear in the context.

A substitution θ is a mapping from a term set T to T such that;

(1) $\theta(f) = f$ if $f \in F$ and $\theta(f) = 0$,

(2) $\theta(f(M_1, \ldots, M_n)) \equiv f(\theta(M_1), \ldots, \theta(M_n))$ if $f(M_1, \ldots, M_n) \in T$.

Thus, for term M, $\theta(M)$ is determined by its values on the variable symbols occurring in M. Following common usage, we write this as $M\theta$ instead of $\theta(M)$.

Consider an extra constant $\square$ called a hole and the set $T(F \cup \{\square\}, V)$. Then $C \in T(F \cup \{\square\}, V)$ is called a context on F. We use the notation $C[\ , \ldots, \ ]$ for the context containing n holes ($n \geq 0$), and if $N_1, \ldots, N_n \in T(F, V)$, then $C[N_1, \ldots, N_n]$ denotes the result of placing $N_1, \ldots, N_n$ in the holes of $C[\ , \ldots, \ ]$ from left to right. In particular, $C[\ ]$ denotes a context containing precisely one hole.

N is called a subterm of $M \equiv C[N]$. Let N be a subterm occurrence of M; then, we write $N \subseteq M$, and if $N \not\equiv M$, then we write $N \subset M$. If N_1 and N_2 are subterm occurences of M having no common symbol occurrences (i.e., $M \equiv C[N_1, N_2]$), then N_1, N_2 are called disjoint (denoted by $N_1 \perp N_2$).

A rewriting rule on T is a pair $\langle M_l, M_r \rangle$ of terms in T such that $M_l \notin V$ and any variable in M_r also occurs in M_l. The notation $\triangleright$ denotes a set of rewriting rules on T and we write $M_l \triangleright M_r$ for $\langle M_l, M_r \rangle \in \triangleright$. A $\rightarrow$redex, or redex, is a term $M_1\theta$, where $M_l \triangleright M_r$, and in this case $M_r\theta$ is called a $\rightarrow$contractum, of $M_l\theta$. The set $\triangleright$ of rewriting rules on T defines a reduction relation $\rightarrow$ on T as follows:

$$M \rightarrow N \text{ iff } M \equiv C[M_l\theta],\ N \equiv C[M_r\theta], \text{ and } M_l \triangleright M_r \quad \text{for some } M_l, M_r, C[\ ], \text{ and } \theta.$$

When we want to specify the redex occurence $A \equiv M_l\theta$ of M in this reduction, write $M \xrightarrow{A} N$.

Definition. A term rewriting system R on T is a reduction system $R = \langle T, \rightarrow \rangle$ such that the reduction relation $\rightarrow$ is defined by a set $\triangleright$ of rewriting rules on T. If R has $M_l \triangleright M_r$, then we write $M_l \triangleright M_r \in R$.

If every variable in term M occurs only once, then M is called linear. We say that R is linear iff for any $M_l \triangleright M_r \in R$, M_l is linear. R is called nonlinear if R is not linear.

Let $M \triangleright N$ and $P \triangleright Q$ be two rules in R. Then the two rules are overlapping iff

(1) if $M \triangleright N$ and $P \triangleright Q$ are different rules, then

$$\exists M' \subseteq M\ (M' \notin V), \exists\theta_1, \exists\theta_2, M'\theta_1 \equiv P\theta_2;$$

(2) if $M \triangleright N$ and $P \triangleright Q$ are the same rule, then
$\exists M' \subset M \ (M' \notin V), \exists \theta_1, \exists \theta_2, M'\theta_1 \equiv P\theta_2$.

Note that in (2) we remove the case $M' \equiv M$ which gives the trivial overlapping. We say that R is ambiguous iff R has overlapping rules. R is called nonambiguous if R is not ambiguous [2],[3].

Note that in this paper there are no limitations on R, thus, R may have nonlinear or ambiguous (i.e., overlapping) rewriting rules [2],[3].

2.3 Direct Sum Systems

Let F_1 and F_2 be disjoint sets of function symbols (i.e., $F_1 \cap F_2 = \phi$), then term rewriting systems R_1 on $T(F_1, V)$ and R_2 on $T(F_2, V)$ are called disjoint. Consider disjoint systems R_1 and R_2 having sets $\underset{1}{\triangleright}$ and $\underset{2}{\triangleright}$ of rewriting rules, respectively, then the direct sum system $R_1 \oplus R_2$ is the term rewriting system on $T(F_1 \cup F_2, V)$ having the set $\underset{1}{\triangleright} \cup \underset{2}{\triangleright}$ of rewriting rules. If R_1 and R_2 are term rewriting systems not satisfying the disjoint requirement for function symbols, then we take isomorphic copies R'_1 and R'_2 by replacing each function symbol f of F_i by f^i $(i = 1, 2)$, and use $R'_1 \oplus R'_2$ instead of $R_1 \oplus R_2$. For this reason, considering the direct sum $R_1 \oplus R_2$, we may assume that R_1 and R_2 are always disjoint, i.e., $F_1 \cap F_2 = \phi$.

Note. The above direct sum is different from Klop's [3]: The direct sum of combinatory reduction systems (in which terms are written in *combinator notation*) is defined as the union of two systems with disjoint constant symbols, but with the same application function symbol. Klop pointed out that his direct sum does not preserve the Church-Rosser property.

It is trivial that if $CR(R_1 \oplus R_2)$ then $CR(R_1)$ and $CR(R_2)$. Hence, in the following sections we shall prove $CR(R_1 \oplus R_2)$, assuming that $CR(R_1)$ and $CR(R_2)$ where $R_1 = \langle T(F_1, V), \underset{1}{\rightarrow} \rangle$, $R_2 = \langle T(F_2, V), \underset{2}{\rightarrow} \rangle$, and $R_1 \oplus R_2 = \langle T(F_1 \cup F_2, V), \rightarrow \rangle$. Note that from here on the notation $\rightarrow$ represents the reduction relation on $R_1 \oplus R_2$.

Definition. A *root* is a mapping from $T(F_1 \cup F_2, V)$ to $F_1 \cup F_2 \cup V$ as follows: For $M \in T(F_1 \cup F_2, V)$,

$$root(M) = \begin{cases} f & \text{if } M \equiv f(M_1, \ldots, M_n), \\ M & \text{if } M \text{ is a constant or a variable.} \end{cases}$$

Definition. Let $M \equiv C[B_1, \ldots, B_n] \in T(F_1 \cup F_2, V)$ and $C \not\equiv \Box$. Then write $M \equiv C[\![B_1, \ldots, B_n]\!]$ if $C[\ , \ldots, \ ]$ is a context on F_a and $\forall i, root(B_i) \in F_b$ $(a, b \in \{1, 2\}$and $a \neq b)$. Then the set $Part(M)$ of the parted terms of $M \in T(F_1 \cup F_2, V)$ is inductively defined as follows:

$$Part(M) = \begin{cases} \{M\} & \text{if } M \in T(F_a, V) \ \ (a = 1 \text{ or } 2), \\ \bigcup_i Part(B_i) \cup \{M\} & \text{if } M \equiv C[\![B_1, \ldots, B_n]\!] \ (n > 0). \end{cases}$$

Definition. For a term $M \in T(F_1 \cup F_2, V)$, the rank $r(M)$ of layers of contexts on F_1 and F_2 in M is inductively defined as follows:

$$r(M) = \begin{cases} 1 & \text{if } M \in T(F_a, V) \ \ (a = 1 \text{ or } 2), \\ max_i\{r(B_i)\} + 1 & \text{if } M \equiv C[\![B_1, \ldots, B_n]\!] \ (n > 0). \end{cases}$$

Example. Let a rewriting rule of R_1 be $f(x) \triangleright f(f(x))$, and let a rewriting rule of R_2 be $g(x, x) \triangleright x$, where $F_1 = \{f\}, F_2 = \{2\}, \rho(f) = 1, \rho(g) = 2$. Consider a term $M_0 \equiv g(f(x), g(f(f(g(x, x))), f(x))) \in T(F_1 \cup F_2, V)$. Note that M_0 has a layer structure of contexts on F_1 and F_2 constructed by $g(\Box, g(\Box, \Box))$ on F_2, $f(x), f(f(\Box)), f(x)$ on F_1, and $g(x, x)$ on F_2 from the outside. Then $Part(M_0) = \{M_0, f(x), f(f(g(x, x))), g(x, x)\}$, $root(M_0) = g$. We can write $M \equiv C[\![f(x), f(f(g(x, x))), f(x)]\!]$ where $C[\ ,\ ,\ ] \equiv g(\Box, g(\Box, \Box))$.

$R_1 \oplus R_2$ has the following reduction;

$$\begin{aligned} & M_0 \equiv g(f(x), g(f(f(g(x, x))), f(x))) \\ \to\ & M_1 \equiv g(f(x), g(f(f(x)), f(x))) \\ \to\ & M_2 \equiv g(f(x), g(f(f(x)), f(f(x)))) \\ \to\ & M_3 \equiv g(f(x), f(f(x))) \\ \to\ & M_4 \equiv g(f(f(x)), f(f(x))) \\ \to\ & M_5 \equiv f(f(x)). \end{aligned}$$

Then $r(M_0) = 3, r(M_1) = r(M_2) = r(M_3) = r(M_4) = 2, r(M_5) = 1$.

Lemma 2.1. If $M \to N$ then $r(M) \geq r(N)$.

Proof. It is easily obtained from the definitions of the direct sum $R_1 \oplus R_2$. $\Box$

3 Preserved Systems

A term $M \in T(F_1 \cup F_2, V)$ has a layer structure of contexts on F_1 and F_2, and this structure is modified through a reduction process in a direct sum system $R_1 \oplus R_2$ on $T(F_1 \cup F_2, V)$. If a reduction $M \to N$ results in the disappearance of some layer between two layers in the term M, then, by putting the two layers together, a new layer structure appears in the term N. If no middle layer in M disappears as a result of any reduction, then we say that the layer structure in M is preserved in the direct sum system. In this section we will show that if two term rewriting systems have the Church-Rosser property, then terms with a certain restriction, viz. that their layer structure is preserved under reductions, also have the Church-Rosser property. Using this result, we will prove our conjecture in section 4.

The set of terms reduced from a term M by a reduction relation $\rightarrow$ is denoted by $G_{\rightarrow}(M) = \{N| \ M \xrightarrow{*} N\}$.

Definition. A term M is root preserved (denoted by $r\text{-}Pre(M)$) iff $root(M) \in F_a \Rightarrow \forall N \in G_{\rightarrow}(M), root(N) \in F_a$, where $a \in \{1, 2\}$.

Now we formalize the concept of preserved layer structure.

Definition. A term $M \equiv C[\![B_1, \ldots, B_n]\!]$ $(n > 0)$ is preserved iff M satisfies two conditions;

(1) $r\text{-}Pre(M)$,

(2) $\forall i, B_i$ is preserved.

We write $Pre(M)$ when M is preserved. Note that, by the definition, if $Pre(M)$, then $\forall N \in G_{\rightarrow}(M), Pre(N)$.

Let $M \xrightarrow{A} N$ and $M \equiv C[\![B_1, \ldots, B_n]\!]$. If the redex occurrence A occurs in some B_j, then we write $M \underset{i}{\rightarrow} N$; otherwise $M \underset{o}{\rightarrow} N$. $\underset{i}{\rightarrow}$ and $\underset{o}{\rightarrow}$ are called an inner and an outer reduction, respectively.

Lemma 3.1. Let $Pre(M)$ and $M \equiv C[\![B_1, \ldots, B_n]\!]$. Then,

(1) $M \underset{i}{\rightarrow} N \Rightarrow N \equiv C[\![C_1, \ldots, C_n]\!]$ where $\forall i, B_i \xrightarrow{\equiv} C_i$;

(2) $M \underset{o}{\rightarrow} N \Rightarrow N \equiv C'[\![B_{i_1}, \ldots, B_{i_p}]\!]$ $(1 \le i_j \le n)$, where $C[\ ,\ldots,\ ]$ and $C'[\ ,\ldots,\ ]$ are contexts on the same set F_a $(a = 1$ or $2)$.

Proof. It is immediately proved from $Pre(M)$ and the definition of $\underset{i}{\rightarrow}$, $\underset{o}{\rightarrow}$. □

We consider the term sequences; $\alpha = \langle A_1, \ldots, A_n\rangle$ and $\beta = \langle B_1, \ldots, B_n\rangle$ where $A_i, B_i \in T$. Then, we write $\alpha \propto \beta$ iff $\forall i, j[A_i \equiv A_j \Rightarrow B_i \equiv B_j]$. We define $\alpha \xrightarrow{*} \beta$ by $\forall i, A_i \xrightarrow{*} B_i$.

We extend the above notations to terms. Let $M \equiv C[\![A_1, \ldots, A_n]\!]$, $N \equiv C[\![B_1, \ldots, B_n]\!]$, $\alpha = \langle A_1, \ldots, A_n\rangle$, $\beta = \langle B_1, \ldots, B_n\rangle$. Then write $M \propto N$ if $\alpha \propto \beta$.

We use the relation $\propto$ to deal with nonlinear rewriting rules. For example, let the reduction $f(A_1, A_2, A_3, A_4) \xrightarrow{*} g(A_1)$ be obtained by using the nonlinear rule $f(x, x, y, y) \triangleright g(x)$. Then, we can obtain the reduction $f(B_1, B_2, B_3, B_4) \xrightarrow{*} g(B_1)$ by the same rule if $\langle A_1, A_2, A_3, A_4 \rangle \propto \langle B_1, B_2, B_3, B_4 \rangle$. This leads us to the following lemma.

Lemma 3.2. Let $Pre(M)$, $M \propto N$. If $M \underset{o}{\to} M'$, then $\exists N', N \underset{o}{\to} N' \wedge M' \propto N'$.

Proof. Let $M \equiv C[\![A_1, \ldots, A_n]\!]$, $N \equiv C[\![B_1, \ldots, B_n]\!]$. Then the left side of the rewriting rule used in $M \underset{o}{\to} M'$ occurs in context $C[\ , \ldots, \ ]$. Since $M \propto N$ we can apply this rule to N in the same way, and obtain $N \underset{o}{\to} N'$. By Lemma 3.1(2), it is clear that $M' \propto N'$. $\square$

Lemma 3.3. Let $Pre(M)$, $M \underset{o}{\to} P$, $M \overset{*}{\underset{i}{\to}} N$, $M \propto N$. Then there is a term Q satisfying the diagram in Figure 2, that is,
$\forall M, N, P \in T[M \overset{*}{\underset{i}{\to}} N \wedge M \overset{*}{\underset{i}{\to}} P \wedge M \propto N \Rightarrow \exists Q \in T, N \overset{*}{\underset{i}{\to}} Q \wedge P \overset{*}{\underset{i}{\to}} Q \wedge P \propto Q]$.

Proof. By Lemma 3.2 we obtain a term Q such that $P \propto Q$ and $N \underset{o}{\to} Q$. Using $M \underset{o}{\to} P$, $M \overset{*}{\underset{i}{\to}} N$ and Lemma 3.1(1), (2), we obtain $P \overset{*}{\underset{i}{\to}} Q$. $\square$

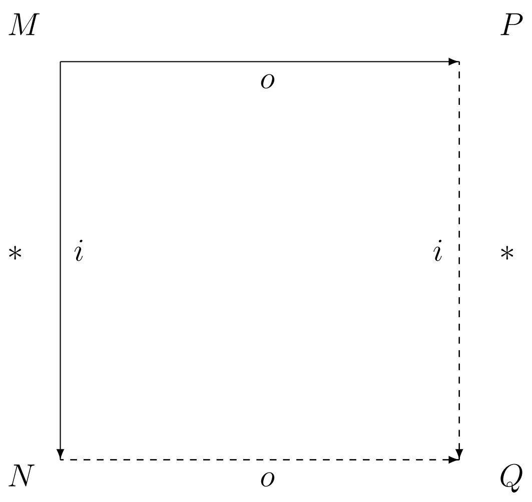

Figure 2

Lemma 3.4. Let $Pre(M)$, $M \overset{*}{\underset{i}{\to}} N$, $M \overset{*}{\underset{o}{\to}} P$, $M \propto N$. Then we can obtain a term Q satisfying Figure 3.

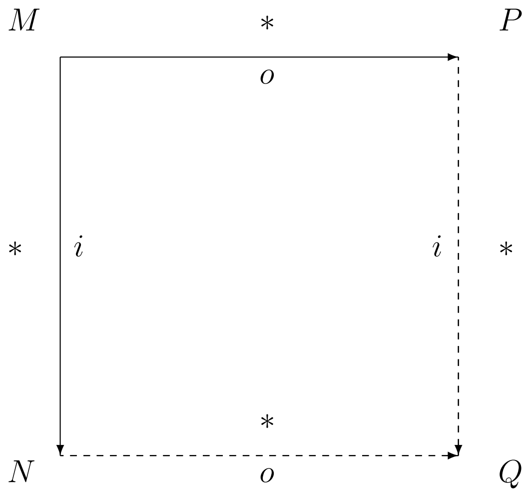

Figure 3

Proof. Using lemma 3.3, the diagram in Figure 4 can be made. □

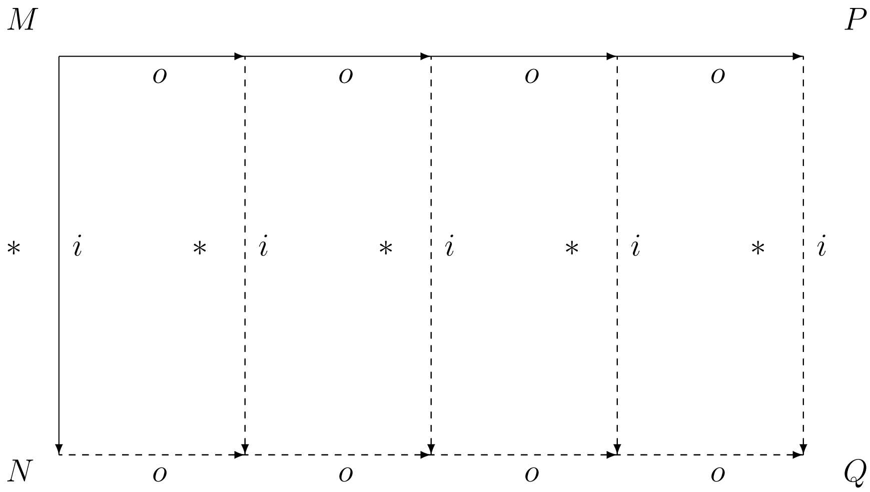

Figure 4

We define the local Church-Rosser property at a term M.

Definition. Let $R = \langle T, \rightarrow \rangle$ be a reduction system and let $M \in T$. Then M is Church-Rosser for $\rightarrow$ (denoted by $CR_{\rightarrow}(M)$ or $CR(M)$) iff
$\forall N, P \in T[M \stackrel{*}{\rightarrow} N \wedge M \stackrel{*}{\rightarrow} P \Rightarrow \exists Q \in T, N \stackrel{*}{\rightarrow} Q \wedge P \stackrel{*}{\rightarrow} Q]$.
Note that $\forall M \in T, CR(M)$ iff $CR(R)$.

We define $M \downarrow N$ by $\exists Q \in T, M \stackrel{*}{\rightarrow} Q \wedge N \stackrel{*}{\rightarrow} Q$.

Lemma 3.5. Let $\alpha = \langle A_1, \ldots, A_n \rangle$ and $\forall i, CR(A_i)$. Then $\exists \beta = \langle B_1, \ \ldots, B_n \rangle [\alpha \xrightarrow{*} \beta \wedge \forall i, j [A_i \downarrow A_j \Rightarrow B_i \equiv B_j]]$.

Proof. Using $CR(A_k)$, it can be shown that $A_i \downarrow A_k \wedge A_k \downarrow A_j \Rightarrow A_i \downarrow A_j$. Hence $\downarrow$ is an equivalence relation and it partitions $\{A_1, \ldots, A_n\}$ in the equivalence class $C_1, \ldots, C_m$. Using the Church-Rosser property for each A_i, we can take a term B_p for each equivalence class $C_p = \{A_{p_1}, \ldots, A_{p_q}\}$ as the diagram in Figure 5. Take $B_{p_1} \equiv, \ldots, \equiv B_{p_q} \equiv B_p$. □

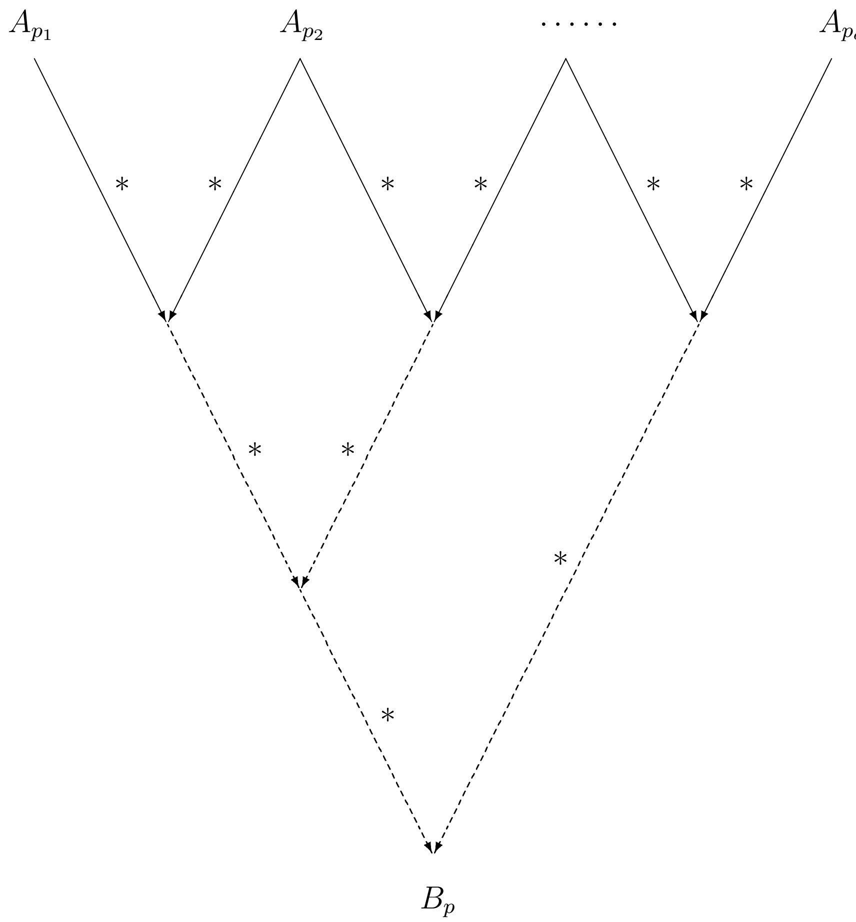

Figure 5

Lemma 3.6. Let $\alpha = \langle A_1, \ldots, A_n \rangle \xrightarrow{*} \beta = \langle B_1, \ldots, B_n \rangle$ and $\forall i, CR(A_i)$. Then $A_i \downarrow A_j$ iff $B_i \downarrow B_j$.

Proof. By the Church-Rosser property for each A_i, it is obvious. □

Lemma 3.7. Let $\alpha = \langle A_1, \ldots, A_n \rangle$, $\forall i, CR(A_i)$, and $\alpha \xrightarrow{*} \beta$, $\alpha \xrightarrow{*} \gamma$. Then we can obtain δ satisfying Figure 6, where $\beta \propto \gamma$ and $\delta \propto \gamma$.

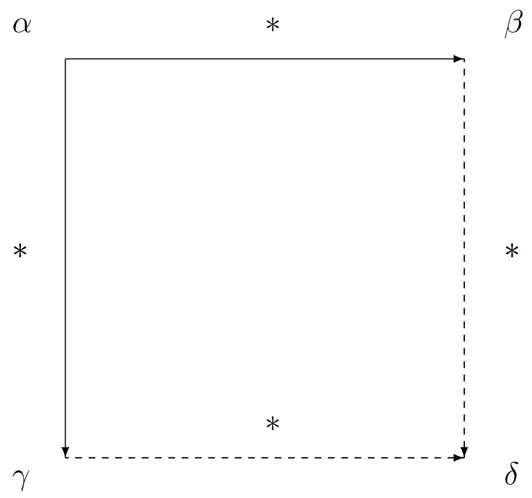

Figure 6

Proof. Let $\beta = \langle B_1, \ldots, B_n \rangle$, $\gamma = \langle C_1, \ldots, C_n \rangle$. By $\forall i, CR(A_i)$, we have a term $\delta' = \langle D'_1, \ldots, D'_n \rangle$ such that $\beta \xrightarrow{*} \delta'$ and $\gamma \xrightarrow{*} \delta'$. Using Lemma 3.5 for δ', we obtain $\delta = \langle D_1, \ldots, D_n \rangle$ such that $\delta' \xrightarrow{*} \delta$ and $D'_i \downarrow D'_j \Rightarrow D_i \downarrow D_j$. Then, by Lemma 3.6, $A_i \downarrow A_j \iff D'_i \downarrow D'_j$, hence $A_i \downarrow A_j \Rightarrow D_i \equiv D_j$. Next we show $\beta \propto \delta$. If $B_i \equiv B_j$, then $A_i \downarrow A_i$, and, thus $D_i \equiv D_j$, hence $\beta \propto \delta$. Similarly we can prove $\gamma \propto \delta$. □

Lemma 3.8. Let $M \equiv C[\![A_1, \ldots, A_n]\!]$, $Pre(M)$, $\forall i, CR(A_i)$. Then we have the diagram in Figure 7, where $N \propto Q$, $P \propto Q$.

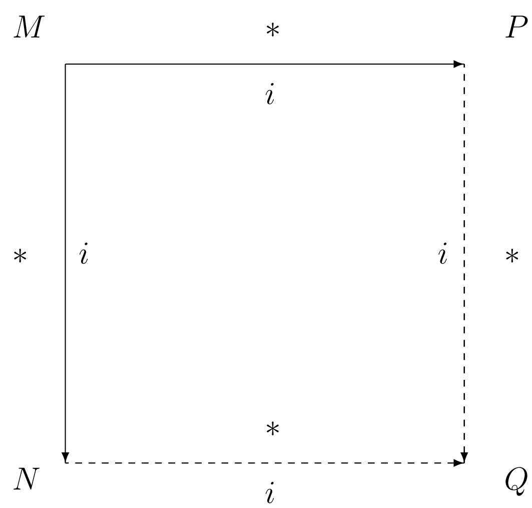

Figure 7

Proof. Since $Pre(M)$, we obtain $N \equiv C[\![B_1, \ldots, B_n]\!], P \equiv C[\![C_1, \ldots, C_n]\!]$, where $\alpha = \langle A_1, \ldots, A_n \rangle \xrightarrow{*} \beta = \langle B_1, \ldots, B_n \rangle$, $\alpha = \langle A_1, \ldots, A_n \rangle \xrightarrow{*} \gamma = \langle C_1, \ldots, C_n \rangle$. Using Lemma 3.7, we can obtain $\delta = \langle D_1, \ldots, D_n \rangle$ such that $\beta \xrightarrow{*} \delta$, $\gamma \xrightarrow{*} \delta$, $\beta \propto \delta$ and $\gamma \propto \delta$. Therefore, take $Q \equiv C[\![D_1, \ldots, D_n]\!]$. □

Lemma 3.9. If $Pre(M)$, then $CR_{\underset{o}{\rightarrow}}(M)$, that is, M is Church-Rosser for $\underset{o}{\rightarrow}$ (Figure 8).

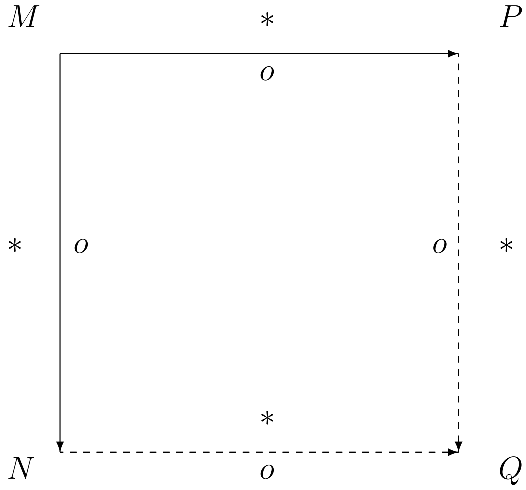

Figure 8

Proof. Let $root(M) \in F_a$ $(a = 1 \text{ or } 2)$. Then, since $Pre(M)$, the outermost part of any term in $G_{\rightarrow}(M)$ is always a context on F_a. Thus $\underset{o}{\rightarrow}$ is determined by only R_a. Hence Church-Rosser for $\underset{o}{\rightarrow}$ is obvious by $CR(R_a)$. □

Theorem 3.1. If $Pre(M)$, then $CR(M)$.

Proof. By induction on the rank $r(M)$ of layers in M. The case $r(M) = 1$ is trivial since $M \in T(F_a, V)$ and $CR(R_a)$ $(a = 1 \text{ or } 2)$; therefore, suppose $r(M) = n > 1$, $M \equiv C[\![A_1, \ldots, A_m]\!]$.

Claim: We obtain the diagram in Figure 9.

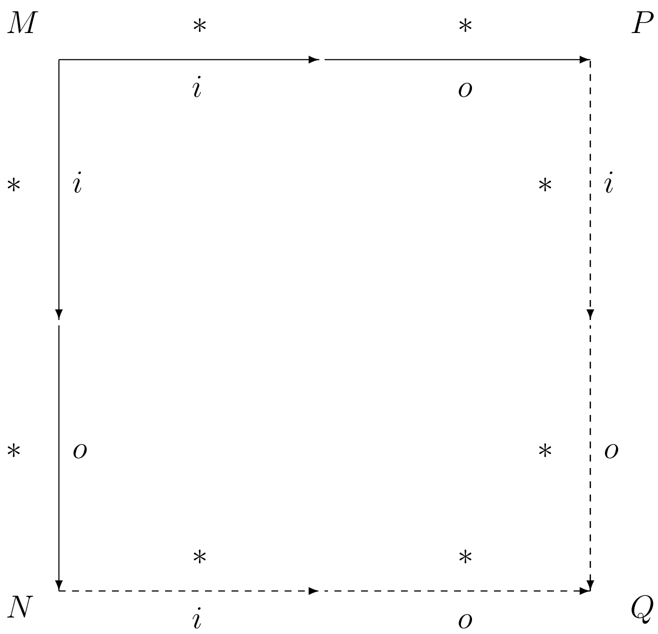

Figure 9

Proof of the claim. By the induction hypothesis, we obtain $\forall i, CR(A_i)$. Using Lemmas 3.8, 3.4 and 3.9 for (1), (2) and (3), respectively, we can obtain the diagram in Figure 10, where $M' \propto Q'$ and $M'' \propto Q'$.

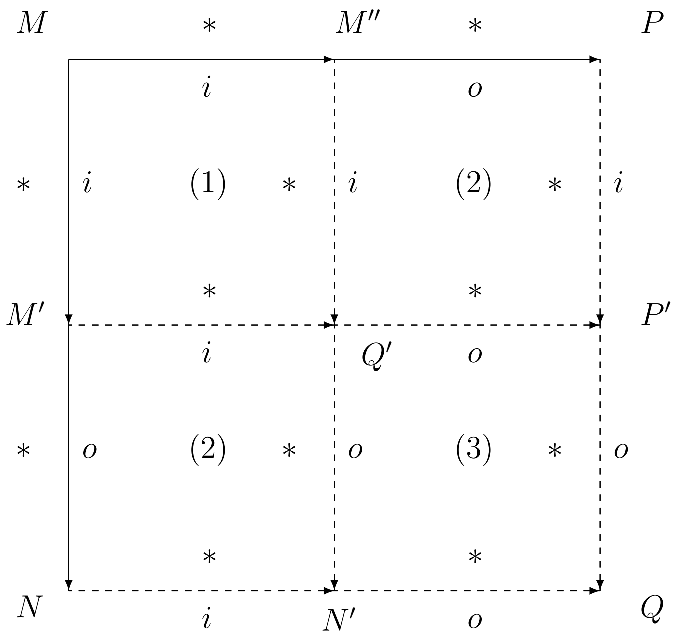

Figure 10

Now we will show $CR(M)$. Note that any reduction $M \xrightarrow{*} M'$ takes the form of $M \xrightarrow[i]{*} \xrightarrow[o]{*} M_1 \xrightarrow[i]{*} \xrightarrow[o]{*} M_2 \xrightarrow[i]{*} \xrightarrow[o]{*} \cdots \xrightarrow[i]{*} \xrightarrow[o]{*} M'$.

Let $M \xrightarrow{*} N$, $M \xrightarrow{*} P$. By splitting $\xrightarrow{*}$ into $\xrightarrow[i]{*} \xrightarrow[o]{*}$ and using the claim, one can draw the diagram in Figure 11. Hence $CR(M)$. $\Box$

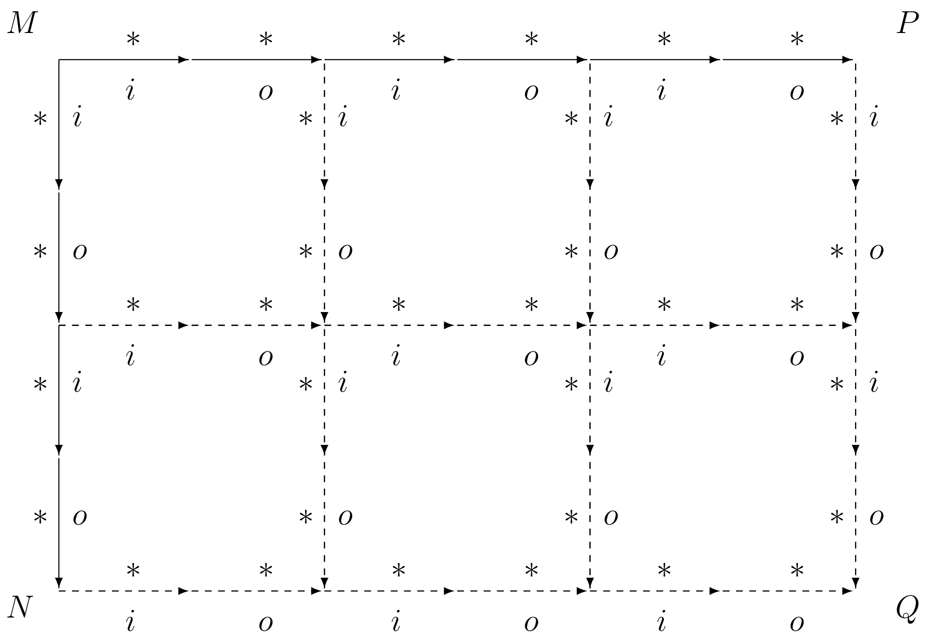

Figure 11

Let $M \xrightarrow{A} N$ where A is a redex occurrence. Then write $M \underset{p}{\rightarrow} N$ if A occurs in a preserved subterm of M, otherwise write $M \underset{np}{\rightarrow} N$.

Theorem 3.2. Let $M \equiv C[\![A_1, \ldots, A_n]\!]$, $\forall i, Pre(A_i)$. Then $CR(M)$.

Proof. If $Pre(M)$, immediate by Theorem 3.1. Hence, suppose $\neg Pre(M)$. Then one can prove the diagrams (1), (2) and (3) in Figure 12, where $M \propto N$ in (1) and $N \propto Q$, $P \propto Q$ in (2), in the same way as for Lemmas 3.4, 3.8 and 3.9, respectively, by replacing $\underset{i}{\rightarrow}$, $\underset{o}{\rightarrow}$ with $\underset{p}{\rightarrow}$, $\underset{np}{\rightarrow}$. Using an analogy to the proof in Theorem 3.1, first, one can obtain the diagram in Figure 13 from the diagrams (1), (2), (3) in Figure 12, and secondly, splitting $\xrightarrow{*}$ into $\underset{p}{\xrightarrow{*}} \underset{np}{\xrightarrow{*}}$, one can show $CR(M)$. $\square$

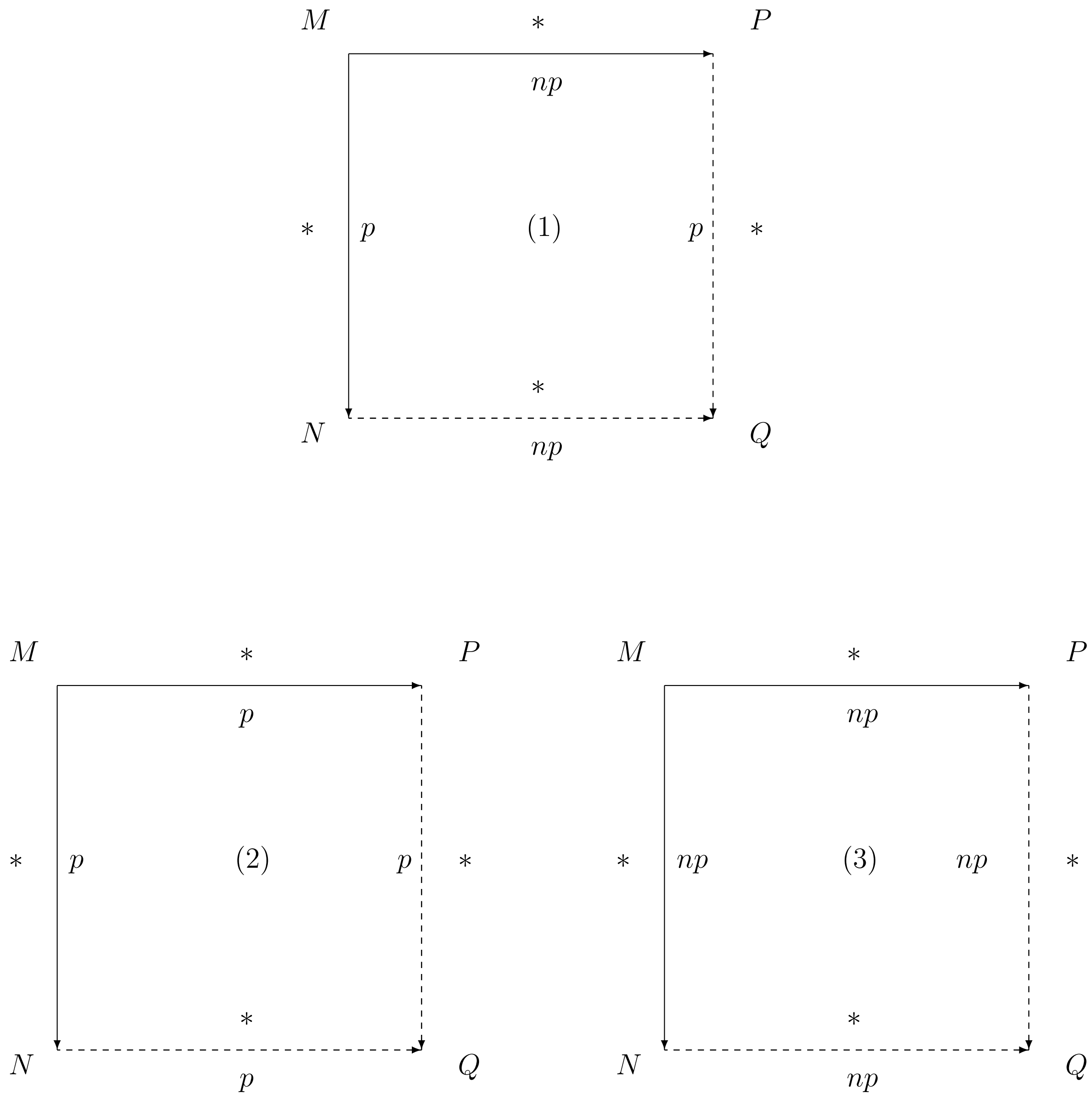

Figure 12

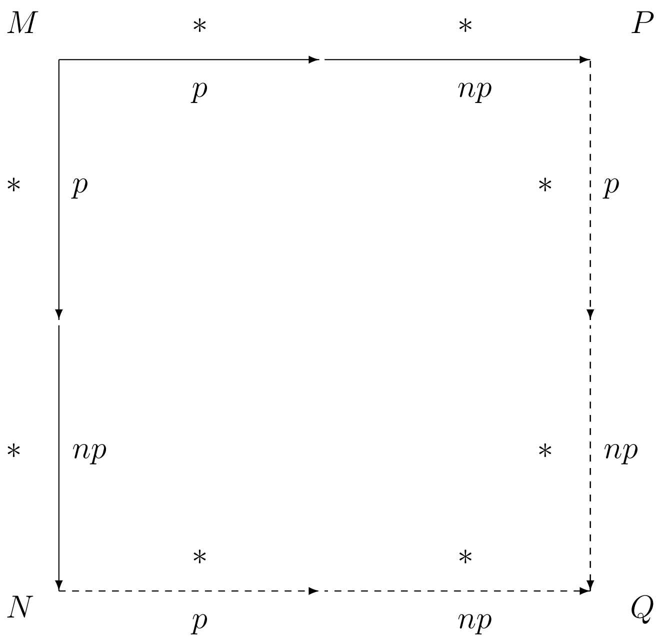

Figure 13

Note. Though $\neg Pre(M)$, the above proof is similar to the proof of Theorem 3.1 in which we assumed $Pre(M)$. This analogy comes from the fact that in Theorem 3.2 a non-preserved context in a term M only occurs at the outermost part of layer structure. However, if some non-preserved context occurs in the middle part, then one cannot prove $CR(M)$ by the analogous method to Theorem 3.1. In the next section we shall consider this case.

4 The Church-Rosser Property for the Direct Sum

In this section we will show that if $CR(R_1)$ and $CR(R_2)$, then $CR(R_1 \oplus R_2)$. This is done by proving $CR(M)$ for any term M by using parallel deletion reduction which deletes the layers of the non-preserve contexts occurring in M. First we shall introduce the following deletion reduction.

Let a term $M \in T(F_1 \cup F_2, V)$ be not preserved. Then there is a term $N \in Part(M)$: $N \equiv \tilde{C}[\![B_1, \ldots, B_n]\!]$, $\neg Pre(N)$, $\forall i, Pre(B_i)$. Since N is not preserved, one has N': $N \xrightarrow{*} N'$, $root(N) \in F_a$, $root(N) \in F_a$ $(a = 1 \text{ or } 2)$. Then the deletion reduction $\underset{d}{\to}$ is defined by replacing N occurring in M by N' as follows:

$M \underset{d}{\to} M' \Rightarrow M \equiv C[N],\ M' \equiv C[N']$,

where N and N' are the above terms.

Then we say N is $\underset{d}{\to}$redex. From this definition, $\underset{d}{\to} \subseteq \xrightarrow{*}$. Let N_1, N_2 be two different $\underset{d}{\rightarrow}$redex occurrences in M, then it is trivial from the definition that N_1, N_2 are disjoint, that is, $N_1 \perp N_2$. Note that $M \in NF_{\underset{d}{\rightarrow}}$ iff $Pre(M)$.

Definition. The maximum depth $d(M)$ of $\underset{d}{\rightarrow}$redex occurrences in M is defined by the following:

$$d(M) = \begin{cases} 0 & \text{if} \quad Pre(M), \\ 1 & \text{if} \quad \neg Pre(M) \text{ and } M \text{ is } \underset{d}{\rightarrow}\text{redex}, \\ max_i\{d(B_i)\} + 1 & \text{if} \quad \neg Pre(M),\ M \text{ is not } \underset{d}{\rightarrow} \text{ redex}, \\ & \quad\quad \text{and } M \equiv C[\![B_1, \ldots, B_n]\!] \quad (n > 0). \end{cases}$$

Lemma 4.1. Let $M \equiv C[B_1, \ldots, B_n]$ and $C \in T(F_a \cup \{\square\}, V)$ $(a = 1 \text{ or } 2)$, then $d(M) \leq max_i\{d(B_i)\} + 1$.

Proof. It is immediately proved from the definition of $d(M)$. □

Lemma 4.2. If $M \rightarrow N$ then $d(M) \geq d(N)$.

Proof. We will prove the lemma by induction on $d(M)$. The case $d(M) \leq 1$ is trivial from the definition. Assume the lemma for $d(M) < k$ $(k > 1)$, then we will show the case $d(M) = k$. Let $M \equiv C[\![B_1, \ldots, B_n]\!]$ $(n > 0)$ and $M \stackrel{A}{\rightarrow} N$.

Case 1. $\exists k, A \subseteq B_k$.

Then $N \equiv C[B_1, \ldots, B_{k-1}, B'_k, B_{k+1}, \ldots, B_n]$ where $B_k \stackrel{A}{\rightarrow} B'_k$. We can obtain $d(B_k) \geq d(B'_k)$ by using the induction hypothesis. Hence by Lemma 4.1,

$$\begin{aligned} d(M) &= max_i\{d(B_i)\} + 1 \\ &\geq max\{d(B_1), \ldots, d(B_{k-1}), d(B'_k), d(B_{k+1}), \ldots, d(B_n)\} + 1 \\ &\geq d(N). \end{aligned}$$

Case 2. Not Case 1.

Then $N \equiv C'[B_{i_1}, \ldots, B_{i_s}]$ where $1 \leq i_j \leq n$ and $C' \in T(F_a \cup \square, V)$ $(a = 1 \text{ or } 2)$. If $s = 0$ then it is clear from $d(N) = 1$ or 0 that $d(M) \leq d(N)$. If $s > 0$ then

$$\begin{aligned} d(M) &= max_i\{d(B_i)\} + 1 \\ &\leq max_j\{d(B_{i_j})\} + 1 \\ &\leq d(N) \end{aligned}$$

for both $C' \not\equiv \square$ and $C' \not\equiv \square$. □

Let $N_1, \ldots, N_n$ be all the $\underset{d}{\rightarrow}$ redex occurrences in M having depth $d(M)$. Note that $N_i \perp N_j$ $(i = j)$. Then the parallel deletion reduction $\underset{pd}{\rightarrow}$ is defined by replacing each $\underset{d}{\rightarrow}$ redex occurrence N_i by N'_i such that $N_i \underset{d}{\rightarrow} N'_i$ at one step, or,

$M \underset{pd}{\rightarrow} N \iff M \equiv C[N_1, \ldots, N_n], N \equiv C[N'_1, \ldots, N'_n]$.

We say that the above $N_1, \ldots, N_n$ are $\underset{pd}{\rightarrow}$ redex occurrences. It is clear that $NF_{\underset{pd}{\rightarrow}} = NF_{\underset{d}{\rightarrow}}$. By the definition of parallel deletion reduction, one can easily prove that if $M \underset{pd}{\rightarrow} M'$ then $d(M) > d(M')$. Hence, every parallel deletion reduction terminates, that is, $SN(\underset{pd}{\rightarrow})$.

Lemma 4.3. Let $M \equiv C[\![A_1, \ldots, A_n]\!] \xrightarrow{M} C[A_{i_1}, \ldots, A_{i_p}]$ where $1 \leq i_j \leq n$, and let $\langle A_1, \ldots, A_n \rangle \propto \langle B_1, \ldots, B_n \rangle$. Then one has a reduction $N \equiv C[B_1, \ldots, B_n] \xrightarrow{N} C'[B_{i_1}, \ldots, B_{i_p}]$.

Proof. The left side of the rewriting rule used in the reduction $\xrightarrow{M}$ occurs in context $C[\ ,\ldots,\ ]$. Hence, one can apply this rewriting rule to N in the same way as for Lemma 3.2. □

Lemma 4.4. Let $d(M) > 1$, $M \equiv C[M_1, \ldots, M_m] \xrightarrow{M} C'[M_{i_1}, \ldots, M_{i_p}]$ $(1 \leq i_j \leq m\ )$, where $M_1, \ldots, M_m$ are all the $\underset{pd}{\rightarrow}$ redex occurrences in M. Let $\langle M_1, \ldots, M_m \rangle \propto \langle M'_1, \ldots, M'_m \rangle$. Then one has a reduction $M' \equiv C[M'_1, \ldots, M'_m] \xrightarrow{M'} C[M'_{i_1}, \ldots, M'_{i_p}]$.

Proof. Let $M \equiv \tilde{C}[\![A_1, \ldots, A_n]\!]$, then $\forall i, \exists j, M_i \subseteq A_j$, and, thus, by replacing each M_i in A_j with M_i, to make A_j, one can obtain $M' \equiv \tilde{C}[A'_1, \ldots, A'_n]$. Now it is evident from $\langle M_1, \ldots, M_m \rangle \propto \langle M'_1, \ldots, M'_m \rangle$, that $\langle A_1, \ldots, A_n \rangle \propto \langle A'_1, \ldots, A'_n \rangle$. Hence Lemma 4.3 applies. □

Lemma 4.5. Let $d(M) > 1$, $M \equiv C[M_1, \ldots, M_m] \xrightarrow{M} C'[M_{i_1}, \ldots, M_{i_p}]$ $(1 \leq i_j \leq m)$, where $M_1, \ldots, M_m$ are all the $\underset{pd}{\rightarrow}$ redex occurrences in M. Let $\langle M_1, \ldots, M_m \rangle \xrightarrow{*} \langle M'_1, \ldots, M'_m \rangle$. Then one can obtain a term sequence $\langle M''_1, \ldots, M''_m \rangle$ such that $\langle M'_1, \ldots, M'_m \rangle \xrightarrow{*} \langle M''_1, \ldots, M''_m \rangle$ and $M' \equiv C[M''_1, \ldots, M''_m] \xrightarrow{M'} C'[M''_{i_1}, \ldots, M''_{i_p}]$.

Proof. In order to prove the lemma by using Lemma 4.4, we only need to find a $\langle M''_1, \ldots, M''_m \rangle$ such that $\langle M_1, \ldots, M_m \rangle \propto \langle M''_1, \ldots, M''_m \rangle$. Since $M_1, \ldots, M_m$ are $\underset{pd}{\rightarrow}$ redex occurrences, we obtain $\forall i, CR(M_i)$ by Theorem 3.2. Therefore, we obtain this $\langle M''_1, \ldots, M''_m \rangle$ by Lemma 3.7, taking $\alpha = \langle M_1, \ldots, M_m \rangle$, $\beta = \gamma = \langle M'_1, \ldots, M'_m \rangle$ and $\delta = \langle M''_1, \ldots, M''_m \rangle$. □

Lemma 4.6. Let $M \to N$, $M \underset{pd}{\to} P$, $d(M) = d(N)$. Then one has the diagram in Figure 14. Note that $d(M) > d(S)$.

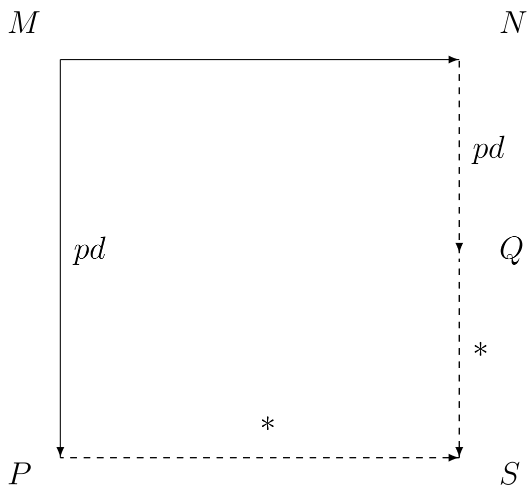

Figure 14

Proof. Let $M \xrightarrow{A} N$. The possible relative positions of the redex occurrence A and all of the $\underset{pd}{\to}$ redex occurrences in M, say $M_1, \ldots, M_m$, are given in the following cases.

Case 1. $\forall i, A \perp M_i$.
Then

$$\begin{aligned} M &\equiv C[M_1, \ldots, M_r, A, M_{r+1}, \ldots, M_m], \\ N &\equiv C[M_1, \ldots, M_r, B, M_{r+1}, \ldots, M_m], \\ P &\equiv C[P_1, \ldots, P_r, A, P_{r+1}, \ldots, P_m], \end{aligned}$$

where $A \underset{A}{\to} B$ and $\forall i, M_i \underset{d}{\to} P_i$. Since all of the $\underset{pd}{\to}$ redex occurrences in N are also $M_1, \ldots, M_m$ (this follows by $d(A) \geq d(B)$; A-contraction cannot create deeper $\underset{d}{\to}$ redex occurrences, in particular no $\underset{pd}{\to}$ redex occurrences), we can take
$Q \equiv C[P_1, \ldots, P_r, B, P_{r+1}, \ldots, P_m]$.
Let $S \equiv Q$, then $P \xrightarrow{*} S$ and $Q \xrightarrow{*} S$.

Case 2. $\exists r, A \subseteq M_r$.
Then

$$\begin{aligned} M &\equiv C[M_1, \ldots, M_{r-1}, M_r, M_{r+1}, \ldots, M_m], \\ N &\equiv C[M_1, \ldots, M_{r-1}, N_r, M_{r+1}, \ldots, M_m], \\ P &\equiv C[P_1, \ldots, P_{r-1}, P_r, P_{r+1}, \ldots, P_m], \end{aligned}$$

where $M_r \xrightarrow{A} N_r$, and $\forall i, M_i \underset{d}{\rightarrow} P_i$. Since each M_i $(i \neq r)$ is also a $\underset{pd}{\rightarrow}$ redex occurrence in N, by using $\underset{pd}{\rightarrow}$ for N, one obtains

$Q \equiv C[P_1, \ldots, P_{r-1}, Q_r, P_{r+1}, \ldots, P_m]$,

where $N_r \overset{\equiv}{\underset{d}{\rightarrow}} Q_r$, whether N_r is a $\underset{pd}{\rightarrow}$ redex occurrence or not (in N). By Theorem 3.2, $CR(M_r)$; therefore, there is a term S_r such that $P_r \xrightarrow{*} S_r$, $Q_r \xrightarrow{*} S_r$. Therefore, take

$S \equiv C[P_1, \ldots, P_{r-1}, S_r, P_{r+1}, \ldots, P_m]$.

Case 3. $\exists j, M_j \subset A$.

Let $M_r, \ldots, M_k$ $(r \leq k)$ be all the $\underset{pd}{\rightarrow}$ redex occurrences in M occurring in A. Then they are also $\underset{pd}{\rightarrow}$ redex occurrences in A. Let $A \equiv D[M_r, \ldots, M_k] \xrightarrow{A} D'[M_{i_1}, \ldots, M_{i_p}]$ $(r \leq i_j \leq k)$.

Then

$$
\begin{aligned}
M &\equiv C[M_1, \ldots, M_{r-1}, D[M_r, \ldots, M_k], M_{k+1}, \ldots, M_m], \\
N &\equiv C[M_1, \ldots, M_{r-1}, D'[M_{i_1}, \ldots, M_{i_p}], M_{k+1}, \ldots, M_m], \\
P &\equiv C[P_1, \ldots, P_{r-1}, D[P_r, \ldots, P_k], P_{k+1}, \ldots, P_m],
\end{aligned}
$$

where $\forall i, M_i \underset{d}{\rightarrow} P_i$. Since $M_1, \ldots, M_{r-1}, M_{k+1}, \ldots, M_m$ are also $\underset{pd}{\rightarrow}$ redex occurrences in N, whether $M_{i_1}, \ldots, M_{i_p}$ are $\underset{pd}{\rightarrow}$ redex occurrences or not (in N), one can obtain

$Q \equiv C[P_1, \ldots, P_{r-1}, D'[Q_{i_1}, \ldots, Q_{i_p}], P_{k+1}, \ldots, P_m]$,

where $\forall j, M_{i_j} \overset{\equiv}{\underset{d}{\rightarrow}} Q_{i_j}$. Now, by using Lemma 4.5, one can show for the subterm $D[P_r, \ldots, P_k]$ in P that there is a sequence $\langle P'_r, \ldots, P'_k \rangle$ such that $\langle P_r, \ldots, P_k \rangle \xrightarrow{*} \langle P'_r, \ldots, P'_k \rangle$ and $D[P'_r, \ldots, P'_k] \rightarrow D'[P'_{i_1}, \ldots, P'_{i_p}]$. Take

$P' \equiv C[P_1, \ldots, P_{r-1}, D'[P'_{i_1}, \ldots, P'_{i_p}], P_{k+1}, \ldots, P_m]$;

then one can have $P \xrightarrow{*} P'$. Since $\forall j, CR(M_{i_j})$, for each j there is S_{i_j} such that $P'_{i_j} \xrightarrow{*} S_{i_j}$, $Q_{i_j} \xrightarrow{*} S_{i_j}$. Therefore, take

$S \equiv C[P_1, \ldots, P_{r-1}, D'[S_{i_1}, \ldots, S_{i_p}], P_{k+1}, \ldots, P_m]$. □

Lemma 4.7. Let $M \rightarrow N$, $M \underset{pd}{\rightarrow} P$, $d(M) > d(N)$, then one has the diagram in Figure 15. Note that $d(M) > d(S)$.

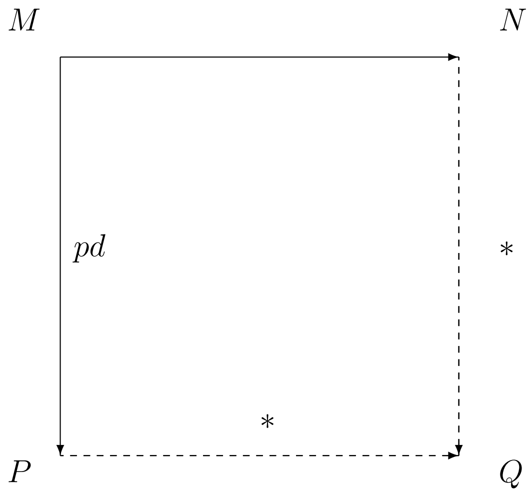

Figure 15

Proof. One can obtain a term S in the same way as for Case 2 and Case 3 in the proof of Lemma 4.6. □

Theorem 4.1. $R_1 \oplus R_2$ has the Church-Rosser property, that is, we have the diagram in Figure 16.

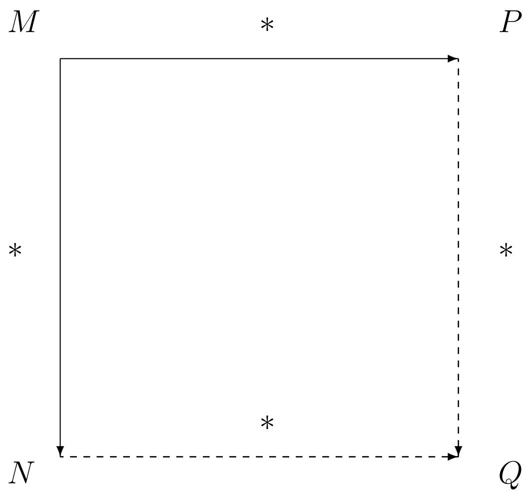

Figure 16

Proof. We will prove $CR(M)$ by induction on $d(M)$. The case $d(M) = 0$ is trivial from Theorem 3.1. Assume $CR(M)$ for $d(M) < n$ $(n > 0)$. Then we will show the following claim.

Claim. One has the diagram in Figure 17 for the case $d(M) \leq n$.

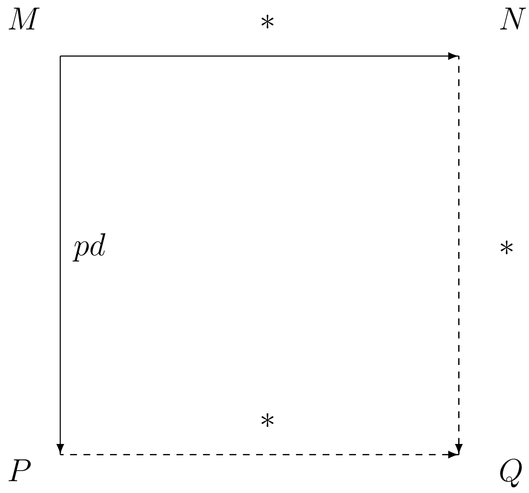

Figure 17

Proof of the Claim. Let $M \stackrel{m}{\to} N$, where $\stackrel{m}{\to}$ denotes a reduction of m ($m \geq 0$) steps. Then we prove the claim by induction on m. The case $m = 0$ is trivial. Assume the claim for $m - 1$ ($m > 0$). We will show the diagram for m. Let $M \to A \stackrel{m-1}{\to} N$.

Case 1. $d(M) = d(A)$. We can obtain the diagram in Figure 18, proving diagram(1) by using Lemma 4.6, diagram(2) by using the induction hypothesis for the claim, and diagram(3) by using the induction hypothesis for the theorem, that is, $CR(B)$, since $d(M) > d(B)$.

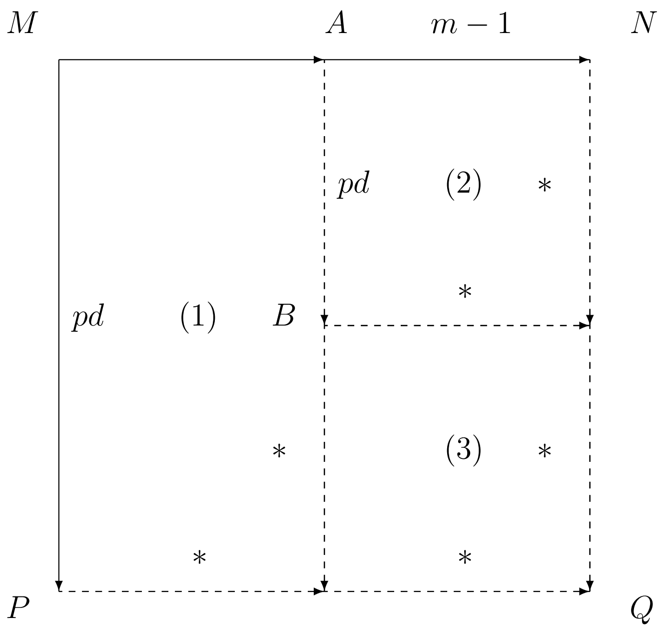

Figure 18

Case 2. $d(M) > d(A)$. We can obtain the diagram in Figure 19, proving diagram(1) by using Lemma 4.7, and diagram(2) by using the induction hypothesis for the theorem, that is, $CR(A)$.

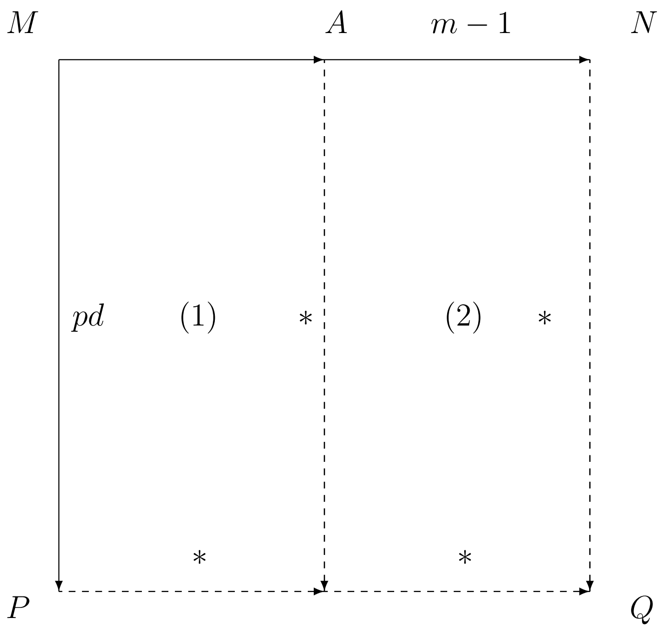

Figure 19

Now we will prove $CR(M)$ for $d(M) = n$. The diagram in Figure 20 can be obtained, where diagram(1) and diagram(2) are shown by the claim and the induction hypothesis, that is, $CR(A)$, respectively. □

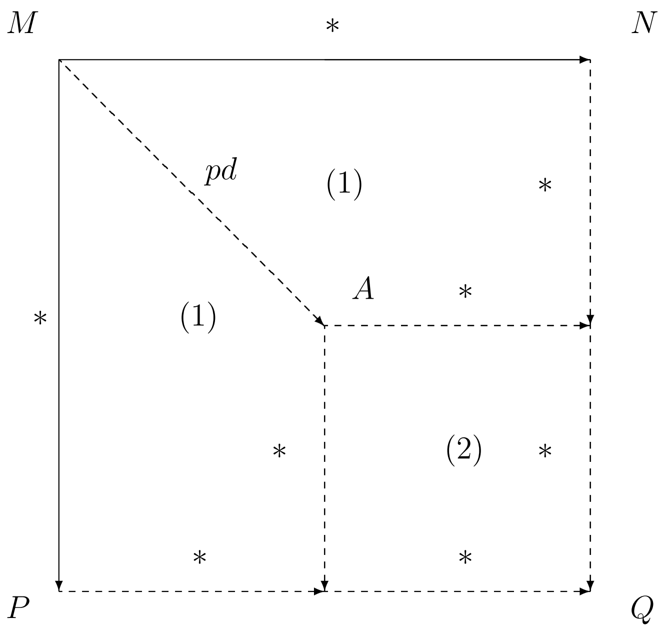

Figure 20

Corollary 4.1. $CR(R_1) \wedge CR(R_2) \iff CR(R_1 \oplus R_2)$.

Proof. $\Leftarrow$ is trivial, and $\Rightarrow$ is proved by Theorem 4.1. □

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