

# Counterexamples to Termination for the Direct Sum of Term Rewriting Systems

Yoshihito TOYAMA

NTT Electrical Communications Laboratories  
3-9-11 Midori-cho, Musashino-shi, Tokyo 180 Japan

## Abstract

The direct sum of two term rewriting systems is the union of systems having disjoint sets of function symbols. It is shown that the direct sum of two term rewriting systems is not terminating, even if these systems are both terminating.

*Keyword:* Term rewriting system, termination

## Introduction

A term rewriting system  $R$  is a set of rewriting rules  $M \rightarrow N$ , where  $M$  and  $N$  are terms [1, 3, 5]. The direct sum system  $R_1 \oplus R_2$  is defined as the union of two term rewriting systems with disjoint function symbols [8]. It was proved [8] that for any term rewriting systems  $R_1$  and  $R_2$ ,

*$R_1 \oplus R_2$  is confluent iff  $R_1$  and  $R_2$  are confluent.*

By replacing *confluent* with *terminating* in the above proposition, the analogous conjecture for the terminating property has the form:

*$R_1 \oplus R_2$  is terminating iff  $R_1$  and  $R_2$  are terminating.*

However, the answer to this conjecture is negative against our expectation. We show the counterexamples to this conjecture and its modifications.

# Counterexamples

A counterexample to the above conjecture is obtained by  $R_1$  and  $R_2$  having the following rewriting rules [8]:

$$R_1 \quad \left\{ F(0, 1, x) \rightarrow F(x, x, x) \right.$$

$$R_2 \quad \left\{ \begin{array}{l} G(x, y) \rightarrow x \\ G(x, y) \rightarrow y \end{array} \right.$$

It is trivial that  $R_1$  and  $R_2$  are terminating. However,  $R_1 \oplus R_2$  is not terminating, because  $R_1 \oplus R_2$  has the infinite reduction sequence:

$$\begin{aligned} F(G(0, 1), G(0, 1), G(0, 1)) &\rightarrow F(0, G(0, 1), G(0, 1)) \rightarrow F(0, 1, G(0, 1)) \\ &\rightarrow F(G(0, 1), G(0, 1), G(0, 1)) \rightarrow \dots \end{aligned}$$

This counterexample also provides a negative answer to the same question for the direct sum of recursive program schemes suggested by Klop [6].

Dershowitz showed the following theorem [1, 2, 3] for terminating of the union system:

*Theorem (Dershowitz 1981). Let  $R_1$  and  $R_2$  be two term rewriting systems. Suppose that  $R_1$  is left linear, and  $R_2$  is right linear, and there is no overlap between the left-hand sides of  $R_1$  and the right-hand sides of  $R_2$ . Then, the union of the two systems is terminating iff both  $R_1$  and  $R_2$  are terminating.*

However, Dershowitz's Theorem [2, 3] is not correct, because the above counterexample refutes his theorem<sup>1</sup>.

In this counterexample, note that  $R_2$  is not confluent. Hence, Toyama conjectured that under the assumption of confluence for  $R_1$  and  $R_2$ ,  $R_1 \oplus R_2$  is terminating iff  $R_1$  and  $R_2$  are terminating [8]. Since the direct sum of two term rewriting systems always preserves their confluence, this conjecture can be stated by the form:

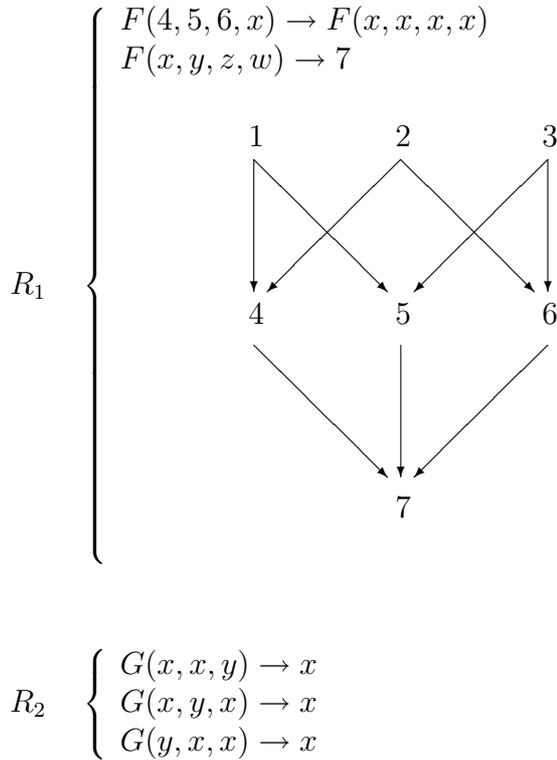
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<sup>1</sup> The version of Dershowitz's Theorem in [1] is correct since the definition of overlap in [1] is different from it in [2, 3]. However, the examples in [1] are wrong, since the definition of overlap is in [2, 3]. This remark is the basis of the letters from Leo Bachmair (on October 24, 1986) and from Nachum Dershowitz (on November 11, 1986).

$R_1 \oplus R_2$  is canonical iff  $R_1$  and  $R_2$  are canonical,

where canonical means confluent and terminating.

However, this conjecture is also not true. Klop and Barendregt showed a counterexample [7] by extending Toyama's counterexample. Consider  $R_1$  and  $R_2$  having the following rewriting rules:



Then,  $R_1$  is confluent, because any term can be reduced into 7.  $R_1$  is also terminating; no term can be reduced into 4, 5, and 6, hence, the first rule cannot be applied infinitely. Thus,  $R_1$  is canonical. Clearly,  $R_2$  is canonical.

However,  $R_1 \oplus R_2$  is not canonical, since  $F(t, t, t, t)$  with  $t \equiv G(1, 2, 3)$  reduces to itself:

$$\begin{aligned}
 &F(t, t, t, t) \rightarrow \dots \rightarrow F(G(4, 4, 3), G(5, 2, 5), G(1, 6, 6), t) \rightarrow \dots \\
 &\rightarrow F(4, 5, 6, t) \rightarrow F(t, t, t, t) \rightarrow \dots
 \end{aligned}$$

We say  $R$  is irreducible if for any rule  $M \rightarrow N$  in  $R$ ,  $M$  and  $N$  are normal forms in  $R - \{M \rightarrow N\}$ . In Klop and Barendregt's counterexample,  $R_1$  is not

irreducible, since the left-hand side  $F(4, 5, 6, x)$  and the right-hand side  $F(x, x, x, x)$  of the first rule can be reduced by using other rules. Hence, Hsiang conjectured [4] that for irreducible term rewriting systems  $R_1$  and  $R_2$ ,  $R_1 \oplus R_2$  is canonical iff  $R_1$  and  $R_2$  are canonical. Clearly, the direct sum of two systems always preserves their irreducibility. Hence, Hsiang's conjecture can be shown in the form:

*$R_1 \oplus R_2$  is canonical and irreducible iff  $R_1$  and  $R_2$  are canonical and irreducible.*

However, Hsiang's conjecture is also not true. We can find the following counterexample to his conjecture by extending Klop and Barendregt's counterexample. Let  $R_1$  and  $R_2$  have the following rewriting rules:

$$R_1 \left\{ \begin{array}{l} F(f_4(x, x), f_5(x, x), f_6(x, x), y, x) \rightarrow F(y, y, y, y, x) \\ F(x, y, z, u, 0) \rightarrow 1 \\ f_1(0, x) \rightarrow f_4(0, x) \\ f_1(x, 0) \rightarrow f_5(x, 0) \\ f_2(0, x) \rightarrow f_4(0, x) \\ f_2(x, 0) \rightarrow f_6(x, 0) \\ f_3(0, x) \rightarrow f_5(0, x) \\ f_3(x, 0) \rightarrow f_6(x, 0) \\ f_4(0, 0) \rightarrow 1 \\ f_5(0, 0) \rightarrow 1 \\ f_6(0, 0) \rightarrow 1 \end{array} \right.$$

$$R_2 \left\{ \begin{array}{l} G(x, x, y) \rightarrow x \\ G(x, y, x) \rightarrow x \\ G(y, x, x) \rightarrow x \end{array} \right.$$

Then, we can show that  $R_1$  and  $R_2$  are canonical and irreducible. However,  $R_1 \oplus R_2$  is not canonical, since  $F(t, t, t, t, 0)$  with  $t \equiv G(f_1(0, 0), f_2(0, 0), f_3(0, 0))$  reduces to itself:

$$\begin{aligned} & F(t, t, t, t, 0) \rightarrow \dots \rightarrow \\ & F(G(f_4(0, 0), f_4(0, 0), f_3(0, 0)), G(f_5(0, 0), f_2(0, 0), f_5(0, 0)), \\ & \quad G(f_1(0, 0), f_6(0, 0), f_6(0, 0)), t, 0) \rightarrow \dots \\ & \rightarrow F(f_4(0, 0), f_5(0, 0), f_6(0, 0), t, 0) \rightarrow F(t, t, t, t, 0) \rightarrow \dots \end{aligned}$$

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