Tree-Sequent Methods for Subintuitionistic Predicate Logics

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Abstract. Subintuitionistic logics are a class of logics defined by using Kripke models with more general conditions than those for intuitionistic logic. In this paper we study predicate logics of this kind by the method of tree-sequent calculus (a special form of Labelled Deductive System). After proving the completeness with respect to some classes of Kripke models, we introduce Hilbert-style axiom systems and prove their completeness through a translation from the tree-sequent calculi. This gives a solution to the problem posed by Restall.

1 Introduction

Subintuitionistic logics are a class of logics defined by using Kripke models with more general conditions than those for intuitionistic logic. They are interpreted in Kripke models similarly to intuitionistic logic, except that some of the conditions (reflexivity, transitivity and persistence) on models are not imposed. The propositional part of these logics has been studied, for example, in [3, 5, 6, 8, 15, 18, 19]. Some of them are considered as the strict implication fragments of the corresponding modal logics, but others such as those in [18] are not.

In [15], Restall described difficulties in extending subintuitionistic logics to the first-order predicate case. One of the difficulties is that if we take the same clauses in the definition of Kripke models as the standard ones for intuitionistic predicate logic then, for instance, the formula $\forall x A(x) \to A(a)$ is no longer valid. This is easily seen in Kripke models whose accessibility relation is not reflexive. So Restall suggested that the domain of quantification should be the domain of the world where the quantified formula is interpreted, and that Kripke models under consideration should have constant domains. However, then proving completeness turned out to be much more complicated than the propositional case, and was left as an open problem.

A solution to axiomatizing a predicate extension of a subintuitionistic logic was given in [20] only for the strict implication fragment of the modal logic $\mathsf{S4}$, where Zimmermann proved the completeness of a Hilbert-style system using a Henkin construction. Note that in the case of the strict implication fragment of $\mathsf{S4}$, one can use a restricted form of the deduction theorem, but it is not clear whether the same method works for other predicate extensions of subintuitionistic logics where the deduction theorem is more restricted.

In this paper, we take a different approach to the problem posed by Restall. We systematically study predicate extensions of several subintuitionistic logics by the method of tree-sequent calculus [12], which is a special form of Labelled Deductive System [7]. This style of formulation is useful in axiomatizing various logics defined through Kripke models without regard to intuitionistic or modal languages (see, e.g. [9, 10, 17]). We prove the completeness of the tree-sequent calculi by constructing a counter model called a saturated tree-sequent. Then we introduce Hilbert-style axiom systems, and define a translation of tree-sequents into formulas to show that each inference rule of the tree-sequent calculi is simulated in the corresponding Hilbert-style systems. This yields the completeness of the Hilbert-style systems with respect to the corresponding classes of Kripke models.

In previous work [10], we presented a tree-sequent calculus for a predicate extension of Visser's propositional logic [18] whose Kripke models have expanding domains. However, no Hilbert-style system for the predicate logic has been obtained. One of the points worth mentioning is that in proving completeness through tree-sequent calculi, models with constant domains are easier to handle than models with expanding domains, while in the usual Henkin construction they are more difficult the other way around.

The organization of the paper is as follows. In Section 2 we define subintuitionistic logics by using Kripke models. In Section 3 we introduce tree-sequent calculi. In Section 4 we prove the completeness of the tree-sequent calculi by a construction of saturated tree-sequents. In Section 5 we introduce Hilbert-style systems and prove the completeness through a translation from the tree-sequent calculi. In Section 6 we conclude with a discussion of related work.

Notation. Our language $\mathcal{L}$ has the following symbols: countably many variables $x_1, x_2, \ldots$; countably many m-ary predicate symbols $p^m_1, p^m_2, \ldots$ for each $m \in \mathbb{N}$; and the logical symbols $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$. (To simplify the argument, we do not consider constants or function symbols.) The set of formulas is constructed from these in the usual way. Parentheses are often omitted using the convention that $\wedge$ and $\vee$ bind more strongly than $\rightarrow$. We use $A, B, C, \ldots$ for formulas.

The symbol $\top$ is defined as an abbreviation of the formula $\bot \rightarrow \bot$. For a finite set Φ of formulas, $\bigwedge \Phi$ (resp. $\bigvee \Phi$) is defined as the conjunction (resp. disjunction) of all formulas in Φ, if Φ is non-empty, otherwise $\top$ (resp. $\bot$).

2 Semantics of Subintuitionistic Logics

Subintuitionistic logics are defined semantically using Kripke models which are more general than models for intuitionistic logic. The propositional part of these logics has been studied in the literature (e.g. [5, 6, 15]) where the least subintuitionistic logic is interpreted in the same Kripke models as those for the modal logic K. The predicate extensions of subintuitionistic logics we consider here are based on Kripke models with constant domains, following the suggestion in [15].

Below we introduce several predicate extensions of subintuitionistic logics as sets of formulas that are valid in some classes of Kripke models.

Definition 1 (Model). Let W be a non-empty set, R be a binary relation on W, and D be a non-empty set. For each m-ary predicate symbol p and each $a \in W$, we suppose a relation $p^{\mathcal{I}(a)}$ on D^m. Then the quadruple $\langle W, R, D, \mathcal{I} \rangle$ is called a *model*.

A model $\langle W, R, D, \mathcal{I} \rangle$ is said to be (i) *reflexive* if for any $a \in W$, aRa, (ii) *transitive* if for any $a, b, c \in W$, aRb and bRc imply aRc, and (iii) *persistent* if for any predicate symbol p and any $a, b \in W$, aRb implies $p^{\mathcal{I}(a)} \subseteq p^{\mathcal{I}(b)}$.

Let $M = \langle W, R, D, \mathcal{I} \rangle$ be a model, and $\mathcal{L}[D]$ be the extended language obtained from $\mathcal{L}$ by adding a constant symbol $\underline{d}$ for each $d \in D$. The relation $\models_M$ between $a \in W$ and each closed formula of $\mathcal{L}[D]$ is defined inductively as follows:

$$
\begin{array}{lll}
a \models_M p(\underline{d_1}, \ldots, \underline{d_m}) & \text{iff} & (d_1, \ldots, d_m) \in p^{\mathcal{I}(a)}; \\
a \not\models_M \bot; & & \\
a \models_M A \wedge B & \text{iff} & a \models_M A \text{ and } a \models_M B; \\
a \models_M A \vee B & \text{iff} & a \models_M A \text{ or } a \models_M B; \\
a \models_M A \to B & \text{iff} & \text{for all } b \in W \text{ with } aRb,\ b \not\models_M A \text{ or } b \models_M B; \\
a \models_M \forall x A & \text{iff} & a \models_M A[\underline{d}/x] \text{ for all } d \in D; \\
a \models_M \exists x A & \text{iff} & a \models_M A[\underline{d}/x] \text{ for some } d \in D.
\end{array}
$$

A formula A is *valid* in a model $M = \langle W, R, D, \mathcal{I} \rangle$, if $a \models_M \forall \overrightarrow{x} A$ for all $a \in W$, where $\overrightarrow{x} = \langle x_1, \ldots, x_n \rangle$ is an enumeration of all free variables in A, and $\forall \overrightarrow{x}$ is an abbreviation for $\forall x_1 \cdots \forall x_n$. ($\forall \overrightarrow{x} A$ is the universal closure of A.)

The predicate logic K^I is defined as the set of all formulas that are valid in every model. Also, the predicate logics KT^I, $\mathsf{K4}^I$, $\mathsf{S4}^I$ and BQL^I are defined as the sets of all formulas that are valid in every reflexive model, transitive model, reflexive and transitive model, and transitive and persistent model, respectively. We use the notation $\mathcal{X}$ to denote any of the logics K^I, KT^I, $\mathsf{K4}^I$, $\mathsf{S4}^I$ and BQL^I.

An example of a formula that is not necessarily valid in the models defined above is $(A \to (B \to C)) \to (B \to (A \to C))$.

The following is an immediate consequence of an induction on the structure of closed formulas.

Lemma 1. *Let $M = \langle W, R, D, \mathcal{I} \rangle$ be a transitive and persistent model. Then, for any closed formula A of $\mathcal{L}[D]$ and any $a, b \in W$, if aRb and $a \models_M A$ then $b \models_M A$.*

Proof. By induction on the structure of A. □

3 Tree-Sequent Calculi

In this section we introduce tree-sequent calculi for subintuitionistic logics as a special form of Labelled Deductive Systems [7], where labels are used for representing a tree-structure and its nodes. In the next sections we prove the soundness and completeness of the tree-sequent calculi with respect to the semantics described in the previous section.

The basic idea of tree-sequent comes from manipulating sequents that reflect the structure of Kripke models directly. For that, each tree-sequent consists of a tree of ordinary sequents.

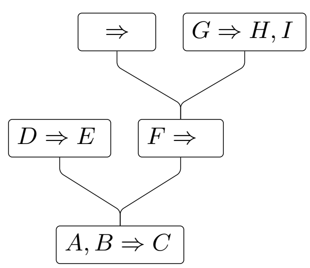
Definition 2 (Tree-sequent). A *label* is a finite sequence of natural numbers $\langle n_1, \ldots, n_m \rangle$. We use letters $\alpha, \beta, \ldots$ for labels. If $\alpha = \langle n_1, \ldots, n_m \rangle$ then $\alpha \cdot n$ denotes the label $\langle n_1, \ldots, n_m, n \rangle$. β is an *immediate successor* (*im-successor*, for short) of α, if $\beta = \alpha \cdot n$ for some natural number n. β is a *successor* of α, if $\beta = ((\cdots((\alpha \cdot n_1) \cdot n_2) \cdots) \cdot n_{m-1}) \cdot n_m$ for some natural numbers $n_1, \ldots, n_m$. A *tree* is a set of labels $\mathcal{T}$ such that $\langle\rangle \in \mathcal{T}$ and for each $\alpha \cdot n \in \mathcal{T}$, $\alpha \in \mathcal{T}$. A *labelled formula* is a pair $\alpha : A$ where α is a label and A is a formula of the language $\mathcal{L}$. A *tree-sequent* is an expression $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ where Γ and Δ are finite sets of labelled formulas, $\mathcal{T}$ is a tree, and each label in Γ, Δ is an element of $\mathcal{T}$.

Thus a tree-sequent $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ represents a finite labelled tree whose structure is induced by $\mathcal{T}$. Each node α of it is associated with a sequent $\Gamma_\alpha \Rightarrow \Delta_\alpha$ where Γ_α (resp. Δ_α) is the set of formulas A such that $\alpha : A \in \Gamma$ (resp. $\alpha : A \in \Delta$).

Example 1. Let $\mathcal{T} = \{\langle\rangle, \langle 1 \rangle, \langle 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}$. The tree-sequent

$$\langle\rangle : A, \langle\rangle : B, \langle 1 \rangle : D, \langle 2 \rangle : F, \langle 2, 2 \rangle : G \overset{\mathcal{T}}{\Rightarrow} \langle\rangle : C, \langle 1 \rangle : E, \langle 2, 2 \rangle : H, \langle 2, 2 \rangle : I$$

represents the following tree of sequents:



To make the argument succinct and precise, we use below representation of sequents with labels rather than the tree form as above. A translation of tree-sequents into formulas of the language $\mathcal{L}$ will be defined in Section 5.

Now we introduce tree-sequent calculi for subintuitionistic logics. These systems define inference schemas to manipulate tree-sequents. First we introduce the most basic system $\mathbf{T}\mathsf{K}^I$, which is the tree-sequent calculus for the logic K^I. The axioms (initial tree-sequents) of $\mathbf{T}\mathsf{K}^I$ are of the following forms:

$$\alpha : A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A \ \text{(Ax)} \qquad \alpha : \bot, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta \ (\bot\text{L})$$

The inference rules of $\mathbf{T}\mathsf{K}^I$ are the following:

$$\frac{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ \text{(Weakening L)} \qquad \frac{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : B}\ \text{(Weakening R)}$$

$$\frac{\alpha : A, \alpha : B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : A \wedge B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ (\wedge\text{L}) \qquad \frac{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A \quad \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : B}{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A \wedge B}\ (\wedge\text{R})$$

$$\frac{\alpha : A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta \quad \alpha : B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : A \vee B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ (\vee\text{L}) \qquad \frac{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A, \alpha : B}{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A \vee B}\ (\vee\text{R})$$

$$\frac{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha \cdot n : A \quad \alpha \cdot n : B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : A \to B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ (\to\text{L})$$

$$\frac{\alpha \cdot n : A, \Gamma \overset{\mathcal{T} \cup \{\alpha \cdot n\}}{\Rightarrow} \Delta, \alpha \cdot n : B}{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A \to B}\ (\to\text{R})^{\ddagger}$$

$$\frac{\alpha : A[y/x], \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : \forall x A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ (\forall\text{L}) \qquad \frac{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A[z/x]}{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : \forall x A}\ (\forall\text{R})_{\text{VC}}$$

$$\frac{\alpha : A[z/x], \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : \exists x A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ (\exists\text{L})_{\text{VC}} \qquad \frac{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A[y/x]}{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : \exists x A}\ (\exists\text{R})$$

where the subscript VC means the eigenvariable condition: z does not occur in the conclusion. The superscript ‡ means the following condition: $\alpha \cdot n$ does not occur in $\mathcal{T}$.

The above tree-sequent calculus $\mathbf{T}\mathsf{K}^I$ has no cut rule. It also dispenses with contraction rules since a tree-sequent consists of finite sets of labelled formulas.

In Fig. 1, we show the rules ($\to$L) and $(\to\text{R})^{\ddagger}$ with representation in the tree form. In the rule ($\to$L), the formula $A \to B$ is introduced in the antecedent of the parent of the node having the formula A or B in the premisses. Similarly, in the rule $(\to\text{R})^{\ddagger}$, the formula $A \to B$ is introduced in the succedent of the parent node. Note that, by the condition of the rule $(\to\text{R})^{\ddagger}$, the node having the sequent $A \Rightarrow B$ in the premiss is *trimmed* in the conclusion.

Example 2. The tree-sequent $\overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle : \forall x(A \to B) \to (A \to \forall x B)$, where x is not free in A, is provable in $\mathbf{T}\mathsf{K}^I$ as follows. Let $\mathcal{T} = \{\langle\rangle, \langle 1\rangle, \langle 1,1\rangle\}$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle 1,1\rangle : A \overset{\mathcal{T}}{\Rightarrow} \langle 1,1\rangle : B, \langle 1,1\rangle : A \qquad \langle 1,1\rangle : B, \langle 1,1\rangle : A \overset{\mathcal{T}}{\Rightarrow} \langle 1,1\rangle : B}{\langle 1\rangle : A \to B, \langle 1,1\rangle : A \overset{\mathcal{T}}{\Rightarrow} \langle 1,1\rangle : B}\ (\to\text{L})}{\langle 1\rangle : \forall x(A \to B), \langle 1,1\rangle : A \overset{\mathcal{T}}{\Rightarrow} \langle 1,1\rangle : B}\ (\forall\text{L})}{\langle 1\rangle : \forall x(A \to B), \langle 1,1\rangle : A \overset{\mathcal{T}}{\Rightarrow} \langle 1,1\rangle : \forall x B}\ (\forall\text{R})_{\text{VC}}}{\langle 1\rangle : \forall x(A \to B) \overset{\{\langle\rangle,\langle 1\rangle\}}{\Rightarrow} \langle 1\rangle : A \to \forall x B}\ (\to\text{R})^{\ddagger}}{\overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle : \forall x(A \to B) \to (A \to \forall x B)}\ (\to\text{R})^{\ddagger}$$

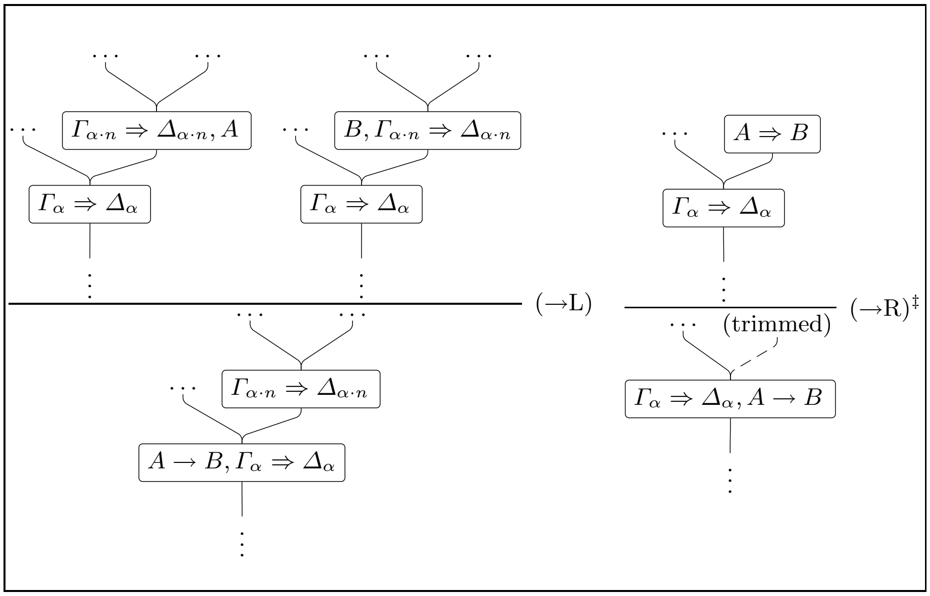


Fig. 1. (→L) and (→R)‡ in the tree form

The tree-sequent formulation works also for logics with modal language by regarding the formula $\top \to A$ of subintuitionistic logics as $\Box A$ of modal logics. For instance, a tree-sequent calculus for a predicate version of the modal logic K may be defined as a system with the following rules for the modal operator:

$$\frac{\alpha \cdot n : A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : \Box A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ (\Box\mathrm{L}) \qquad \frac{\Gamma \overset{\mathcal{T}\cup\{\alpha\cdot n\}}{\Rightarrow} \Delta, \alpha \cdot n : A}{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : \Box A}\ (\Box\mathrm{R})^{\ddagger}$$

The completeness theorem of the tree-sequent calculus for K is proved similarly to that for $\mathbf{T}\mathsf{K}^I$ in the next section.

Next we introduce tree-sequent calculi for the logics KT^I, $\mathsf{K4}^I$, $\mathsf{S4}^I$ and BQL^I defined in the previous section. The tree-sequent calculus $\mathbf{T}\mathsf{KT}^I$ is obtained from $\mathbf{T}\mathsf{K}^I$ by adding the following rule:

$$\frac{\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A \quad \alpha : B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : A \to B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ (\to\mathrm{L}_{\,\mathrm{Ref}})$$

The tree-sequent calculus $\mathbf{T}\mathsf{K4}^I$ is obtained from $\mathbf{T}\mathsf{K}^I$ by adding the following rule:

$$\frac{\alpha \cdot n : A \to B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : A \to B, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}\ (\to\mathrm{L}_{\,\mathrm{Tran}})$$

The tree-sequent calculus $\mathbf{T}\mathsf{S4}^I$ is obtained from $\mathbf{T}\mathsf{K}^I$ by adding both of the rules $(\to\mathrm{L}_{\,\mathrm{Ref}})$ and $(\to\mathrm{L}_{\,\mathrm{Tran}})$. Finally, the tree-sequent calculus $\mathbf{T}\mathsf{BQL}^I$ is obtained from $\mathbf{T}\mathsf{K}^I$ by adding the following rule:

$$\frac{\alpha \cdot n : A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta}{\alpha : A, \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta} \text{ (Drop)}$$

In the sequel, we use the notation $\mathbf{T}\mathcal{X}$ to denote any of the tree-sequent calculi $\mathbf{T}\mathsf{K}^I$, $\mathbf{T}\mathsf{KT}^I$, $\mathbf{T}\mathsf{K4}^I$, $\mathbf{T}\mathsf{S4}^I$ and $\mathbf{T}\mathsf{BQL}^I$.

4 Completeness of Tree-Sequent Calculi

In this section we show that the tree-sequent calculi introduced in the previous section are sufficient to prove all formulas that are valid in the respective classes of Kripke models. For this purpose we construct a counter model for any formula that is not provable in each tree-sequent calculus. The converse directions, i.e., all provable formulas are valid, are shown through the soundness of Hilbert-style systems in the next section.

In the following, Γ, Δ and $\mathcal{T}$ are possibly infinite in the expression $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ of a tree-sequent. In the case where Γ, Δ and $\mathcal{T}$ are all finite, the tree-sequent $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ is said to be finite. A (possibly infinite) tree-sequent $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ is provable in $\mathbf{T}\mathcal{X}$, if $\Gamma' \overset{\mathcal{T}'}{\Rightarrow} \Delta'$ is provable in $\mathbf{T}\mathcal{X}$ for some finite tree-sequent $\Gamma' \overset{\mathcal{T}'}{\Rightarrow} \Delta'$ such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ and $\mathcal{T}' \subseteq \mathcal{T}$.

Definition 3 ($\mathbf{T}\mathcal{X}$-saturatedness). A tree-sequent $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ is $\mathbf{T}\mathcal{X}$*-saturated*, if it satisfies the following conditions:

- $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ is not provable in $\mathbf{T}\mathcal{X}$;
- (For $\mathbf{T}\mathsf{BQL}^I$ only) If $\alpha : A \in \Gamma$, then $\beta : A \in \Gamma$ for every successor β of α;
- For any $\alpha \in \mathcal{T}$,
  - [($\wedge$L)] If $\alpha : A \wedge B \in \Gamma$, then $\alpha : A \in \Gamma$ and $\alpha : B \in \Gamma$;
  - [($\wedge$R)] If $\alpha : A \wedge B \in \Delta$, then $\alpha : A \in \Delta$ or $\alpha : B \in \Delta$;
  - [($\vee$L)] If $\alpha : A \vee B \in \Gamma$, then $\alpha : A \in \Gamma$ or $\alpha : B \in \Gamma$;
  - [($\vee$R)] If $\alpha : A \vee B \in \Delta$, then $\alpha : A \in \Delta$ and $\alpha : B \in \Delta$;
  - [($\to$L)] If $\alpha : A \to B \in \Gamma$, then $\alpha \cdot n : A \in \Delta$ or $\alpha \cdot n : B \in \Gamma$ for every im-successor $\alpha \cdot n$ of α;
  - [($\to$L$_{\text{Ref}}$)] (For $\mathbf{T}\mathsf{KT}^I/\mathbf{T}\mathsf{S4}^I$ only) If $\alpha : A \to B \in \Gamma$, then $\alpha : A \in \Delta$ or $\alpha : B \in \Gamma$;
  - [($\to$L$_{\text{Tran}}$)] (For $\mathbf{T}\mathsf{K4}^I/\mathbf{T}\mathsf{S4}^I$ only) If $\alpha : A \to B \in \Gamma$, then $\beta : A \to B \in \Gamma$ for every successor β of α;
  - [($\to$R)] If $\alpha : A \to B \in \Delta$, then there exists an im-successor $\alpha \cdot n$ of α such that $\alpha \cdot n : A \in \Gamma$ and $\alpha \cdot n : B \in \Delta$;
  - [($\forall$L)] If $\alpha : \forall x A \in \Gamma$, then $\alpha : A[y/x] \in \Gamma$ for every variable y;
  - [($\forall$R)] If $\alpha : \forall x A \in \Delta$, then $\alpha : A[z/x] \in \Delta$ for some variable z;
  - [($\exists$L)] If $\alpha : \exists x A \in \Gamma$, then $\alpha : A[z/x] \in \Gamma$ for some variable z;
  - [($\exists$R)] If $\alpha : \exists x A \in \Delta$, then $\alpha : A[y/x] \in \Delta$ for every variable y.

The next lemma shows that any unprovable tree-sequent extends to a $\mathbf{T}\mathcal{X}$-saturated tree-sequent, which represents a counter model to the tree-sequent. The process of this extension differs from the usual Henkin-style construction in that it handles the whole counter model rather than separate nodes of a model. The structure of tree-sequents is thus useful in proving completeness of various logics defined through Kripke models.

Lemma 2. *If a finite tree-sequent $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ is not provable in $\mathbf{T}\mathcal{X}$, then there exists a $\mathbf{T}\mathcal{X}$-saturated tree-sequent $\Gamma^+ \overset{\mathcal{T}^+}{\Rightarrow} \Delta^+$ such that $\Gamma \subseteq \Gamma^+$, $\Delta \subseteq \Delta^+$ and $\mathcal{T} \subseteq \mathcal{T}^+$.*

Proof. Suppose that a finite tree-sequent $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ is not provable in $\mathbf{T}\mathcal{X}$. In the following, we construct an infinite sequence of finite tree-sequents $\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1$, $\Gamma^2 \overset{\mathcal{T}^2}{\Rightarrow} \Delta^2, \ldots$ and obtain $\Gamma^+ \overset{\mathcal{T}^+}{\Rightarrow} \Delta^+$ as the union of them.

Let $B_1, B_2, \ldots$ be an enumeration of all formulas of the language $\mathcal{L}$ such that each formula appears infinitely many times. We also enumerate all variables as $x_1, x_2, \ldots$.

Let $\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1 \equiv \Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$. The i-th step, which is the step of extension from $\Gamma^i \overset{\mathcal{T}^i}{\Rightarrow} \Delta^i$ to $\Gamma^{i+1} \overset{\mathcal{T}^{i+1}}{\Rightarrow} \Delta^{i+1}$, consists of operations for the formula B_i. In these operations, unprovability of the tree-sequent is preserved. The operations executed in the i-th step are as follows:

- (For $\mathbf{T}\mathsf{BQL}^I$ only) For each $\alpha \in \mathcal{T}^i$, if $\alpha : B_i \in \Gamma^i$, then add $\beta : B_i$ to Γ^i for every successor β of α. Unprovability is preserved because of the inference rule (Drop).
- According to the form of B_i, one of the following operations is executed for each label $\alpha \in \mathcal{T}^i$.
  - [B_i is of the form $C \wedge D$]
    If $\alpha : B_i \in \Gamma^i$, then add $\alpha : C$ and $\alpha : D$ to Γ^i. Unprovability is preserved because of the inference rule ($\wedge$L).
    If $\alpha : B_i \in \Delta^i$, then add $\alpha : C$ or $\alpha : D$ to Γ^i, so that unprovability is preserved because of the inference rule ($\wedge$R).
  - [B_i is of the form $C \vee D$]
    Symmetric to the case for $C \wedge D$.
  - [B_i is of the form $C \to D$]
    (For $\mathbf{T}\mathsf{K}^I/\mathbf{T}\mathsf{BQL}^I$) If $\alpha : B_i \in \Gamma^i$, then add $\alpha \cdot n : C$ to Δ^i or $\alpha \cdot n : D$ to Γ^i for each im-successor $\alpha \cdot n$ of α, so that unprovability is preserved.
    (For $\mathbf{T}\mathsf{KT}^I$) If $\alpha : B_i \in \Gamma^i$, then add $\alpha : C$ to Δ^i or $\alpha : D$ to Γ^i, and add $\alpha \cdot n : C$ to Δ^i or $\alpha \cdot n : D$ to Γ^i for each im-successor $\alpha \cdot n$ of α, so that unprovability is preserved.
    (For $\mathbf{T}\mathsf{K4}^I$) If $\alpha : B_i \in \Gamma^i$, then add $\beta : B_i$ to Γ^i for every successor β of α, and add $\alpha \cdot n : C$ to Δ^i or $\alpha \cdot n : D$ to Γ^i for each im-successor $\alpha \cdot n$ of α, so that unprovability is preserved.
    (For $\mathbf{T}\mathsf{S4}^I$) If $\alpha : B_i \in \Gamma^i$, then add $\beta : B_i$ to Γ^i for every successor β of α, add $\alpha : C$ to Δ^i or $\alpha : D$ to Γ^i, and add $\alpha \cdot n : C$ to Δ^i or $\alpha \cdot n : D$ to Γ^i for each im-successor $\alpha \cdot n$ of α, so that unprovability is preserved.

(For $\mathbf{T}\mathcal{X}$) If $\alpha : B_i \in \Delta^i$, then add a new im-successor $\alpha \cdot n$ of α to $\mathcal{T}^i$, add $\alpha \cdot n : C$ to Γ^i, and add $\alpha \cdot n : D$ to Δ^i. Unprovability is preserved because of the inference rule $(\to\mathrm{R})^\ddagger$.

- [B_i is of the form $\forall x C$]
  If $\alpha : B_i \in \Gamma^i$, then add $\alpha : C[x_1/x], \ldots, \alpha : C[x_i/x]$ to Γ^i. Unprovability is preserved because of the inference rule $(\forall\mathrm{L})$.
  If $\alpha : B_i \in \Delta^i$, then take a fresh variable z, and add $\alpha : C[z/x]$ to Δ^i. Unprovability is preserved because of the inference rule $(\forall\mathrm{R})_{\mathrm{VC}}$.
- [B_i is of the form $\exists x C$]
  Symmetric to the case for $\forall x C$.

Now let $\Gamma^+ \overset{\mathcal{T}^+}{\Rightarrow} \Delta^+ \equiv \bigcup_{n=1}^{\infty} \Gamma^n \overset{\bigcup_{n=1}^{\infty} \mathcal{T}^n}{\Rightarrow} \bigcup_{n=1}^{\infty} \Delta^n$. It is easy to verify that the tree-sequent $\Gamma^+ \overset{\mathcal{T}^+}{\Rightarrow} \Delta^+$ is $\mathbf{T}\mathcal{X}$-saturated. □

We are now ready to prove the completeness of the tree-sequent calculi.

Theorem 1 (Completeness of $\mathbf{T}\mathcal{X}$). *For any formula A, if $A \in \mathcal{X}$ then the tree-sequent $\overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle : A$ is provable in $\mathbf{T}\mathcal{X}$.*

Proof. Suppose that $\overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle : A$ is not provable in $\mathbf{T}\mathcal{X}$. Then by Lemma 2, there exists a $\mathbf{T}\mathcal{X}$-saturated tree-sequent $\Gamma^+ \overset{\mathcal{T}^+}{\Rightarrow} \Delta^+$ such that $\{\langle\rangle : A\} \subseteq \Delta^+$. Now we define a model $M = \langle W, R, D, \mathcal{I}\rangle$ as follows:

– W is $\mathcal{T}^+$;
– (For $\mathbf{T}\mathsf{K}^I$) $\alpha R \beta$ iff β is an im-successor of α in $\mathcal{T}^+$;
(For $\mathbf{T}\mathsf{KT}^I$) $\alpha R \beta$ iff $\alpha = \beta$ or β is an im-successor of α in $\mathcal{T}^+$;
(For $\mathbf{T}\mathsf{K4}^I/\mathbf{T}\mathsf{BQL}^I$) $\alpha R \beta$ iff β is a successor of α in $\mathcal{T}^+$;
(For $\mathbf{T}\mathsf{S4}^I$) $\alpha R \beta$ iff $\alpha = \beta$ or β is a successor of α in $\mathcal{T}^+$;
– D is the set of all variables;
– $(y_1, \ldots, y_m) \in p^{\mathcal{I}(\alpha)}$ iff $\alpha : p(y_1, \ldots, y_m) \in \Gamma^+$.

It is easy to verify that M satisfies the conditions of respective models.

Now we show by induction on B that for any labelled formula $\alpha : B$,

– if $\alpha : B \in \Gamma^+$ then $\alpha \models_M B[\underline{\overrightarrow{x_B}}/\overrightarrow{x_B}]$, and
– if $\alpha : B \in \Delta^+$ then $\alpha \not\models_M B[\underline{\overrightarrow{x_B}}/\overrightarrow{x_B}]$,

where $\overrightarrow{x_B}$ is an enumeration of all free variables in B. Here we consider only the cases where B is of the form $C \to D$ and of the form $\forall x C$.

- [B is of the form $C \to D$]
  (For $\mathbf{T}\mathsf{K}^I$) If $\alpha : B \in \Gamma^+$ then by the $\mathbf{T}\mathsf{K}^I$-saturatedness, for any im-successor $\alpha \cdot n$ of α, we have $\alpha \cdot n : C \in \Delta^+$ or $\alpha \cdot n : D \in \Gamma^+$. By the induction hypothesis, we have $\alpha \cdot n \not\models_M C[\underline{\overrightarrow{x_C}}/\overrightarrow{x_C}]$ or $\alpha \cdot n \models_M D[\underline{\overrightarrow{x_D}}/\overrightarrow{x_D}]$. Hence, $\alpha \models_M (C \to D)[\underline{\overrightarrow{x_{C\to D}}}/\overrightarrow{x_{C\to D}}]$.
  (For $\mathbf{T}\mathsf{KT}^I$) If $\alpha : B \in \Gamma^+$ then by the $\mathbf{T}\mathsf{KT}^I$-saturatedness, $\alpha : C \in \Delta^+$ or $\alpha : D \in \Gamma^+$, and for any im-successor $\alpha \cdot n$ of α, we have $\alpha \cdot n : C \in \Delta^+$ or

$\alpha \cdot n : D \in \Gamma^+$. By the induction hypothesis, we have $\alpha \not\models_M C[\underline{\overrightarrow{x_C}}/\overrightarrow{x_C}]$ or $\alpha \models_M D[\underline{\overrightarrow{x_D}}/\overrightarrow{x_D}]$, and $\alpha \cdot n \not\models_M C[\underline{\overrightarrow{x_C}}/\overrightarrow{x_C}]$ or $\alpha \cdot n \models_M D[\underline{\overrightarrow{x_D}}/\overrightarrow{x_D}]$. Hence, $\alpha \models_M (C \to D)[\underline{\overrightarrow{x_{C\to D}}}/\overrightarrow{x_{C\to D}}]$.
(For $\mathbf{T}\mathsf{K4}^I/\mathbf{T}\mathsf{BQL}^I$) If $\alpha : B \in \Gamma^+$ then by the $\mathbf{T}\mathsf{K4}^I/\mathbf{T}\mathsf{BQL}^I$-saturatedness, for any successor β of α, we have $\beta : B \in \Gamma^+$, and $\beta : C \in \Delta^+$ or $\beta : D \in \Gamma^+$. By the induction hypothesis, we have $\beta \not\models_M C[\underline{\overrightarrow{x_C}}/\overrightarrow{x_C}]$ or $\beta \models_M D[\underline{\overrightarrow{x_D}}/\overrightarrow{x_D}]$. Hence, $\alpha \models_M (C \to D)[\underline{\overrightarrow{x_{C\to D}}}/\overrightarrow{x_{C\to D}}]$.
(For $\mathbf{T}\mathsf{S4}^I$) If $\alpha : B \in \Gamma^+$ then by the $\mathbf{T}\mathsf{S4}^I$-saturatedness, $\alpha : C \in \Delta^+$ or $\alpha : D \in \Gamma^+$, and for any successor β of α, we have $\beta : B \in \Gamma^+$, and $\beta : C \in \Delta^+$ or $\beta : D \in \Gamma^+$. By the induction hypothesis, we have $\alpha \not\models_M C[\underline{\overrightarrow{x_C}}/\overrightarrow{x_C}]$ or $\alpha \models_M D[\underline{\overrightarrow{x_D}}/\overrightarrow{x_D}]$, and $\beta \not\models_M C[\underline{\overrightarrow{x_C}}/\overrightarrow{x_C}]$ or $\beta \models_M D[\underline{\overrightarrow{x_D}}/\overrightarrow{x_D}]$. Hence, $\alpha \models_M (C \to D)[\underline{\overrightarrow{x_{C\to D}}}/\overrightarrow{x_{C\to D}}]$.
(For $\mathbf{T}\mathcal{X}$) If $\alpha : B \in \Delta^+$ then by the $\mathbf{T}\mathcal{X}$-saturatedness, there exists an im-successor $\alpha \cdot n$ of α such that $\alpha \cdot n : C \in \Gamma^+$ and $\alpha \cdot n : D \in \Delta^+$. By the induction hypothesis, we have $\alpha \cdot n \models_M C[\underline{\overrightarrow{x_C}}/\overrightarrow{x_C}]$ and $\alpha \cdot n \not\models_M D[\underline{\overrightarrow{x_D}}/\overrightarrow{x_D}]$. Hence, $\alpha \not\models_M (C \to D)[\underline{\overrightarrow{x_{C\to D}}}/\overrightarrow{x_{C\to D}}]$.

- [B is of the form $\forall x C$]
  If $\alpha : B \in \Gamma^+$ then by the $\mathbf{T}\mathcal{X}$-saturatedness, $\alpha : C[y/x] \in \Gamma^+$ for any variable y. By the induction hypothesis, $\alpha \models_M C[y/x][\underline{\overrightarrow{x_{C[y/x]}}}/\overrightarrow{x_{C[y/x]}}]$, which means $\alpha \models_M C[\underline{\overrightarrow{x_C \backslash x}}/\overrightarrow{x_C \backslash x}][\underline{y}/x]$. Hence, we have $\alpha \models_M \forall x C[\underline{\overrightarrow{x_B}}/\overrightarrow{x_B}]$, i.e., $\alpha \models_M (\forall x C)[\underline{\overrightarrow{x_B}}/\overrightarrow{x_B}]$.
  The case where $\alpha : B \in \Delta^+$ is proved similarly.

Since $\langle\rangle : A \in \Delta^+$, we have $\langle\rangle \not\models_M A[\underline{\overrightarrow{x_A}}/\overrightarrow{x_A}]$. Hence, A is not valid in this model M. □

5 Completeness of Hilbert-Style Systems

In this section we introduce Hilbert-style systems for the predicate extensions of subintuitionistic logics, and investigate their relationships with the tree-sequent calculi. Hilbert-style systems for subintuitionistic propositional logics have been studied in [5, 6, 15]. Here we define a translation of tree-sequents into formulas, and show that each inference rule of the tree-sequent calculi is simulated in the corresponding Hilbert-style systems. This yields the completeness of the Hilbert-style systems with respect to the classes of Kripke models.

First we introduce the system $\mathbf{H}\mathsf{K}^I$, which is a Hilbert-style system for the least subintuitionistic logic K^I. The axioms of $\mathbf{H}\mathsf{K}^I$ are the following:

(A1) $A \to A$,

(A2) $(A \to B) \wedge (B \to C) \to (A \to C)$,

(A3) $A \wedge B \to A$,

(A4) $A \wedge B \to B$,

(A5) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$, (A6) $A \to A \vee B$,

(A7) $B \to A \vee B$,

(A8) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$,

(A9) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$,

(A10) $\bot \to A$,

(A11) $\forall x A \to A[y/x]$,

(A12) $A[y/x] \to \exists x A$,

(A13) $\forall x(A \vee B) \to \forall x A \vee B$ where x is not free in B,

(A14) $\exists x A \wedge B \to \exists x(A \wedge B)$ where x is not free in B,

(A15) $\forall x(B \to A) \to (B \to \forall x A)$ where x is not free in B,

(A16) $\forall x(A \to B) \to (\exists x A \to B)$ where x is not free in B.

The inference rules of $\mathbf{H}\mathsf{K}^I$ are the following:

$$\frac{A \quad A \to B}{B}\ (\mathrm{MP}) \qquad \frac{A}{B \to A}\ (\mathrm{AF}) \qquad \frac{A \quad B}{A \wedge B}\ (\wedge\mathrm{I}) \qquad \frac{A}{\forall x A}\ (\mathrm{GR})$$

Hilbert-style systems for the logics KT^I, $\mathsf{K4}^I$, $\mathsf{S4}^I$ and BQL^I are introduced as follows. The system $\mathbf{H}\mathsf{KT}^I$ is obtained from $\mathbf{H}\mathsf{K}^I$ by adding the following axiom:

(A17) $A \wedge (A \to B) \to B$.

The system $\mathbf{H}\mathsf{K4}^I$ is obtained from $\mathbf{H}\mathsf{K}^I$ by adding the following axiom:

(A18) $(A \to B) \to (C \to (A \to B))$.

The system $\mathbf{H}\mathsf{S4}^I$ is obtained from $\mathbf{H}\mathsf{K}^I$ by adding both of the axioms (A17) and (A18). It is the same in provability as the system of [20], but has more natural axioms on $\vee$ and $\exists$. Finally, the system $\mathbf{H}\mathsf{BQL}^I$ is obtained from $\mathbf{H}\mathsf{K}^I$ by adding the following axiom:

(A19) $A \to (B \to A)$.

In the sequel, we use the notation $\mathbf{H}\mathcal{X}$ to denote any of the Hilbert-style systems $\mathbf{H}\mathsf{K}^I$, $\mathbf{H}\mathsf{KT}^I$, $\mathbf{H}\mathsf{K4}^I$, $\mathbf{H}\mathsf{S4}^I$ and $\mathbf{H}\mathsf{BQL}^I$.

Theorem 2 (Soundness of $\mathbf{H}\mathcal{X}$). *For any formula A, if A is provable in $\mathbf{H}\mathcal{X}$ then $A \in \mathcal{X}$.*

Proof. By induction on the proof of A in $\mathbf{H}\mathcal{X}$. □

In the rest of this section, we prove the completeness of the Hilbert-style systems. The difficulty doing this lies in that the deduction theorem is not available in these systems. So we provide below some derivable rules and formulas instead.

Lemma 3. *The following rules are derivable in* $\mathbf{H}\mathcal{X}$:

$$\frac{A \to B \quad B \to C}{A \to C}\ (\mathrm{Tr}) \qquad \frac{A \to B \quad A \to C}{A \to B \wedge C}\ (\to\wedge\mathrm{I})$$

$$\frac{A \to (B \to C) \quad A \to (C \to D)}{A \to (B \to D)}\ (\to\mathrm{Tr})$$

$$\frac{B \to C \quad A \to (C \to D)}{A \to (B \to D)}\ (\mathrm{Tr}\,2) \qquad \frac{A \to (B \to C) \quad C \to D}{A \to (B \to D)}\ (\mathrm{Tr}\,3)$$

$$\frac{A \to B}{(B \to C) \to (A \to C)}\ (\mathrm{Suff}) \qquad \frac{A \to B}{(C \to A) \to (C \to B)}\ (\mathrm{Pref})$$

Proof. See Appendix A. □

Note 1. These rules are used in an inductive proof of the equivalent replacement, which justifies such an expression as $\bigwedge \Gamma$ with the associativity and commutativity of $\wedge$ on provability in $\mathbf{H}\mathcal{X}$.

Lemma 4. *The following formulas are provable in* $\mathbf{H}\mathcal{X}$:

1. $(A \to B) \to (A \vee C \to B \vee C)$,
2. $A \wedge (B \vee C) \to B \vee (A \wedge C)$,
3. $(A \vee B) \wedge (A \vee C) \to A \vee (B \wedge C)$.

Proof. See Appendix A. □

Lemma 5. *The following rules are derivable in* $\mathbf{H}\mathcal{X}$:

$$\frac{A}{C \to D \vee A}\ (\mathrm{R1}) \qquad \frac{A_1 \to A}{(C \to D \vee A_1) \to (C \to D \vee A)}\ (\mathrm{R2})$$

$$\frac{A_1 \wedge A_2 \to A}{(C \to D \vee A_1) \wedge (C \to D \vee A_2) \to (C \to D \vee A)}\ (\mathrm{R3})$$

Proof. See Appendix A. □

Next we define a translation of tree-sequents into formulas. (In the following, tree-sequents are all finite.)

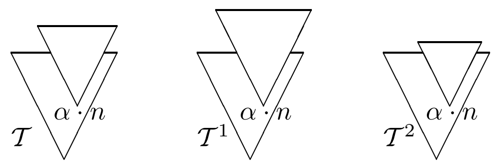
Definition 4 (Formulaic translation). Let $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ be a tree-sequent. For each $\alpha \in \mathcal{T}$, the formula $[\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_\alpha$ is defined inductively as follows:

$$[\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_\alpha \equiv (\bigwedge \Gamma_\alpha) \to (\bigvee \Delta_\alpha) \vee [\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha\cdot n_1} \vee \cdots \vee [\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha\cdot n_k}$$

where $\{\alpha\cdot n_1, \ldots, \alpha\cdot n_k\}$ is the set of im-successors of α in $\mathcal{T}$, and Γ_α (resp. Δ_α) is the set of formulas A with $\alpha : A \in \Gamma$ (resp. $\alpha : A \in \Delta$). Then the *formulaic translation* of $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$ is defined as the universal closure of $[\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\langle\rangle}$.

Now our aim is to show that for any tree-sequent that is provable in one of the tree-sequent calculi, its formulaic translation is provable in the corresponding Hilbert-style system. To facilitate the proof, we give some lemmas.

Lemma 6. *Let* $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$, $\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1$ *and* $\Gamma^2 \overset{\mathcal{T}^2}{\Rightarrow} \Delta^2$ *be tree-sequents that have the same structure except for the parts associated with* $\alpha \cdot n \in \mathcal{T} \cap \mathcal{T}^1 \cap \mathcal{T}^2$ *and all of its successors.*



1. *If* $[\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha \cdot n}$ *is provable in* $\mathbf{H}\mathcal{X}$, *then* $[\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha}$ *is provable in* $\mathbf{H}\mathcal{X}$.
2. *If* $[\![\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1]\!]_{\alpha \cdot n} \rightarrow [\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha \cdot n}$ *is provable in* $\mathbf{H}\mathcal{X}$, *then* $[\![\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1]\!]_{\alpha} \rightarrow [\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha}$ *is provable in* $\mathbf{H}\mathcal{X}$.
3. *If* $[\![\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1]\!]_{\alpha \cdot n} \wedge [\![\Gamma^2 \overset{\mathcal{T}^2}{\Rightarrow} \Delta^2]\!]_{\alpha \cdot n} \rightarrow [\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha \cdot n}$ *is provable in* $\mathbf{H}\mathcal{X}$, *then* $[\![\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1]\!]_{\alpha} \wedge [\![\Gamma^2 \overset{\mathcal{T}^2}{\Rightarrow} \Delta^2]\!]_{\alpha} \rightarrow [\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha}$ *is provable in* $\mathbf{H}\mathcal{X}$.

Proof. Direct application of Lemma 5. □

Lemma 7. *Let* $\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta$, $\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1$ *and* $\Gamma^2 \overset{\mathcal{T}^2}{\Rightarrow} \Delta^2$ *be tree-sequents that have the same structure except for the parts associated with* $\alpha \in \mathcal{T} \cap \mathcal{T}^1 \cap \mathcal{T}^2$ *and all of its successors.*

1. *Suppose that* $[\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha}$ *is provable in* $\mathbf{H}\mathcal{X}$. *Then,* $[\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\langle\rangle}$ *is provable in* $\mathbf{H}\mathcal{X}$.
2. *Suppose that* $[\![\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1]\!]_{\alpha} \rightarrow [\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha}$ *is provable in* $\mathbf{H}\mathcal{X}$. *Then, if* $[\![\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1]\!]_{\langle\rangle}$ *is provable in* $\mathbf{H}\mathcal{X}$ *then* $[\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\langle\rangle}$ *is provable in* $\mathbf{H}\mathcal{X}$.
3. *Suppose that* $[\![\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1]\!]_{\alpha} \wedge [\![\Gamma^2 \overset{\mathcal{T}^2}{\Rightarrow} \Delta^2]\!]_{\alpha} \rightarrow [\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\alpha}$ *is provable in* $\mathbf{H}\mathcal{X}$. *Then, if* $[\![\Gamma^1 \overset{\mathcal{T}^1}{\Rightarrow} \Delta^1]\!]_{\langle\rangle}$ *and* $[\![\Gamma^2 \overset{\mathcal{T}^2}{\Rightarrow} \Delta^2]\!]_{\langle\rangle}$ *are provable in* $\mathbf{H}\mathcal{X}$ *then* $[\![\Gamma \overset{\mathcal{T}}{\Rightarrow} \Delta]\!]_{\langle\rangle}$ *is provable in* $\mathbf{H}\mathcal{X}$.

Proof. Easily seen by applying Lemma 6 as many times as the length of α. □

Lemma 8. *The following formulas are provable in* $\mathbf{H}\mathcal{X}$:

1. $A \wedge C \rightarrow D \vee A$,
2. $\bot \wedge C \rightarrow D$,
3. $(C \rightarrow D) \rightarrow (A \wedge C \rightarrow D)$,
4. $(C \rightarrow D) \rightarrow (C \rightarrow D \vee B)$,
5. $(C \rightarrow D \vee A) \wedge (C \rightarrow D \vee B) \rightarrow (C \rightarrow D \vee (A \wedge B))$,

6. $(A \wedge C \to D) \wedge (B \wedge C \to D) \to ((A \vee B) \wedge C \to D)$,

7. $(C \to D \vee (E \to F \vee A)) \wedge (C \to D \vee (B \wedge E \to F)) \to ((A \to B) \wedge C \to D \vee (E \to F))$,

8. $(A[y/x] \wedge C \to D) \to (\forall x A \wedge C \to D)$,

9. $(C \to D \vee A[y/x]) \to (C \to D \vee \exists x A)$.

The following formula is provable in $\mathbf{H}\mathsf{KT}^I/\mathbf{H}\mathsf{S4}^I$:

10. $(C \to D \vee A) \wedge (B \wedge C \to D) \to ((A \to B) \wedge C \to D)$.

The following formula is provable in $\mathbf{H}\mathsf{K4}^I/\mathbf{H}\mathsf{S4}^I$:

11. $(C \to D \vee ((A \to B) \wedge E \to F)) \to ((A \to B) \wedge C \to D \vee (E \to F))$.

The following formula is provable in $\mathbf{H}\mathsf{BQL}^I$:

12. $(C \to D \vee (A \wedge E \to F)) \to (A \wedge C \to D \vee (E \to F))$.

Proof. See Appendix A. □

Now we prove a crucial lemma for the completeness of the Hilbert-style systems.

Lemma 9. *For any tree-sequent* $\Gamma \stackrel{\mathcal{T}}{\Rightarrow} \Delta$*, if* $\Gamma \stackrel{\mathcal{T}}{\Rightarrow} \Delta$ *is provable in* $\mathbf{T}\mathcal{X}$*, then the formulaic translation of* $\Gamma \stackrel{\mathcal{T}}{\Rightarrow} \Delta$ *is provable in* $\mathbf{H}\mathcal{X}$.

Proof. By induction on the proof of $\Gamma \stackrel{\mathcal{T}}{\Rightarrow} \Delta$ in $\mathbf{T}\mathcal{X}$. All cases except the rules $(\forall \mathrm{R})_{\mathrm{VC}}$ and $(\exists \mathrm{L})_{\mathrm{VC}}$ immediately follow from Lemmas 7 and 8. Here we consider the case where the last applied rule is $(\forall \mathrm{R})_{\mathrm{VC}}$. Then by the induction hypothesis, the universal closure of $\forall z [\![\Gamma \stackrel{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A[z/x]]\!]_{\langle\rangle}$ is provable in $\mathbf{H}\mathcal{X}$, where

$$[\![\Gamma \stackrel{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A[z/x]]\!]_{\langle\rangle} \equiv (\bigwedge \Gamma_{\langle\rangle}) \to (\bigvee \Delta_{\langle\rangle}) \vee [\![\Gamma \stackrel{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A[z/x]]\!]_{\langle n_1\rangle} \vee \cdots \vee [\![\Gamma \stackrel{\mathcal{T}}{\Rightarrow} \Delta, \alpha : A[z/x]]\!]_{\langle n_k\rangle}$$

and $\{\langle n_1\rangle, \ldots, \langle n_k\rangle\}$ is the set of im-successors of $\langle\rangle$ in $\mathcal{T}$. Now, using (A15), (A13) and (Tr 3) repeatedly, we can move $\forall z$ to the front of $A[z/x]$ at the node α. Hence the universal closure of $[\![\Gamma \stackrel{\mathcal{T}}{\Rightarrow} \Delta, \alpha : \forall z A[z/x]]\!]_{\langle\rangle}$ is provable in $\mathbf{H}\mathcal{X}$.

The case where the last applied rule is $(\exists \mathrm{L})_{\mathrm{VC}}$ is proved similarly using also (A16) and (A14). □

We can now prove the completeness of the Hilbert-style systems as well as the soundness of the tree-sequent calculi.

Theorem 3 (Completeness of $\mathbf{H}\mathcal{X}$, Soundness of $\mathbf{T}\mathcal{X}$). *For any formula* A*, the following are equivalent*:

1. $A \in \mathcal{X}$,
2. A *is provable in* $\mathbf{H}\mathcal{X}$,
3. $\stackrel{\{\langle\rangle\}}{\Rightarrow} \langle\rangle : A$ *is provable in* $\mathbf{T}\mathcal{X}$.

Proof. ($1 \Rightarrow 3$) This follows from Theorem 1.
($3 \Rightarrow 2$) Suppose that the tree-sequent $\overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle : A$ is provable in $\mathbf{T}\mathcal{X}$. Then by Lemma 9, the universal closure of $[\![\overset{\{\langle\rangle\}}{\Rightarrow} \langle\rangle : A]\!]_{\langle\rangle}$ is provable in $\mathbf{H}\mathcal{X}$, and hence $\top \to A$ is provable in $\mathbf{H}\mathcal{X}$. Therefore, A is provable in $\mathbf{H}\mathcal{X}$.
($2 \Rightarrow 1$) This follows from Theorem 2. □

6 Conclusion

We have studied predicate extensions of subintuitionistic logics by the method of tree-sequent calculus. First we introduced tree-sequent calculi and proved their completeness with respect to the semantics of subintuitionistic logics. Then we defined a translation of tree-sequents into formulas of the intuitionistic language and proved the completeness of new Hilbert-style systems using the translation.

It seems difficult to axiomatize predicate extensions of subintuitionistic logics in the style of traditional sequent calculus as found in [11, 13, 14]. There are also algebraic approaches to subintuitionistic logics using properties of distributive lattices [1, 4, 16]. However, those studies are limited to the propositional case, and completeness for predicate extensions of subintuitionistic logics has not been obtained by algebraic methods. On the other hand, our approach based on structured sequents can produce requirements, like Lemmas 5 and 8, for completeness of formal systems for various propositional and predicate logics defined through Kripke models. It might be possible to use other formalisms with structured sequents, e.g. display logic [2, 19], for axiomatizing predicate extensions of subintuitionistic logics (see also discussions in [8]). Making precise connections between such formalisms and ours is to be investigated and left as future work.

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A Omitted Proofs

In this appendix we supply some proofs that are omitted in the main body.

Lemma 3. *The following rules are derivable in* $\mathbf{H}\mathcal{X}$:

$$\frac{A \to B \quad B \to C}{A \to C}\ (\mathrm{Tr}) \qquad \frac{A \to B \quad A \to C}{A \to B \wedge C}\ (\to\wedge \mathrm{I})$$

$$\frac{A \to (B \to C) \quad A \to (C \to D)}{A \to (B \to D)}\ (\to \mathrm{Tr})$$

$$\frac{B \to C \quad A \to (C \to D)}{A \to (B \to D)}\ (\mathrm{Tr}\,2) \qquad \frac{A \to (B \to C) \quad C \to D}{A \to (B \to D)}\ (\mathrm{Tr}\,3)$$

$$\frac{A \to B}{(B \to C) \to (A \to C)}\ (\mathrm{Suff}) \qquad \frac{A \to B}{(C \to A) \to (C \to B)}\ (\mathrm{Pref})$$

Proof. For (Tr), we have the following proof.

$$\frac{\dfrac{A \to B \qquad B \to C}{(A \to B) \wedge (B \to C)}\ (\wedge \mathrm{I}) \qquad \dfrac{}{(A \to B) \wedge (B \to C) \to (A \to C)}\ (\mathrm{A2})}{A \to C}\ (\mathrm{MP})$$

For $(\to\wedge \mathrm{I})$, we have the following proof.

$$\frac{\dfrac{A \to B \qquad A \to C}{(A \to B) \wedge (A \to C)}\ (\wedge \mathrm{I}) \qquad \dfrac{}{(A \to B) \wedge (A \to C) \to (A \to B \wedge C)}\ (\mathrm{A5})}{A \to B \wedge C}\ (\mathrm{MP})$$

For $(\to \mathrm{Tr})$, we have the following proof.

$$\frac{\dfrac{A \to (B \to C) \quad A \to (C \to D)}{A \to (B \to C) \wedge (C \to D)}\ (\to\wedge \mathrm{I}) \qquad \dfrac{}{(B \to C) \wedge (C \to D) \to (B \to D)}\ (\mathrm{A2})}{A \to (B \to D)}\ (\mathrm{Tr})$$

For (Tr 2), we have the following proof.

$$\frac{\dfrac{B \to C}{A \to (B \to C)}\ (\mathrm{AF}) \qquad A \to (C \to D)}{A \to (B \to D)}\ (\to \mathrm{Tr})$$

For (Tr 3), we have the following proof.

$$\frac{A \to (B \to C) \qquad \dfrac{C \to D}{A \to (C \to D)}\ (\mathrm{AF})}{A \to (B \to D)}\ (\to \mathrm{Tr})$$

For (Suff), we have the following proof.

$$\dfrac{A \to B \quad \overset{\text{(A1)}}{(B \to C) \to (B \to C)}}{(B \to C) \to (A \to C)}\ (\text{Tr}\,2)$$

For (Pref), we have the following proof.

$$\dfrac{\overset{\text{(A1)}}{(C \to A) \to (C \to A)} \quad A \to B}{(C \to A) \to (C \to B)}\ (\text{Tr}\,3)$$

□

Lemma 4. *The following formulas are provable in* $\mathbf{H}\mathcal{X}$:

1. $(A \to B) \to (A \vee C \to B \vee C)$,
2. $A \wedge (B \vee C) \to B \vee (A \wedge C)$,
3. $(A \vee B) \wedge (A \vee C) \to A \vee (B \wedge C)$.

Proof. 1. We have the following proof.

$$\dfrac{\dfrac{\overset{\text{(A6)}}{B \to B \vee C}}{(A \to B) \to (A \to B \vee C)}\ (\text{Pref}) \quad \dfrac{\overset{\text{(A7)}}{C \to B \vee C}}{(A \to B) \to (C \to B \vee C)}\ (\text{AF})}{(A \to B) \to (A \to B \vee C) \wedge (C \to B \vee C)}\ (\to\!\wedge \text{I})$$

From this and $(A \to B \vee C) \wedge (C \to B \vee C) \to (A \vee C \to B \vee C)$, which is an instance of (A8), we obtain $(A \to B) \to (A \vee C \to B \vee C)$ by (Tr).

2. We have the following proof.

$$\dfrac{\overset{\text{(A4)}}{A \wedge B \to B} \quad \overset{\text{Lemma 4 (1)}}{(A \wedge B \to B) \to ((A \wedge B) \vee (A \wedge C) \to B \vee (A \wedge C))}}{(A \wedge B) \vee (A \wedge C) \to B \vee (A \wedge C)}\ (\text{MP})$$

From this and $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$, which is an instance of (A9), we obtain $A \wedge (B \vee C) \to B \vee (A \wedge C)$ by (Tr).

3. We have the following proof.

$$\dfrac{\dfrac{\overset{\text{(A3)}}{A \wedge (A \vee C) \to A} \quad \overset{\text{(A6)}}{A \to A \vee (B \wedge C)}}{A \wedge (A \vee C) \to A \vee (B \wedge C)}\ (\text{Tr}) \quad \overset{\text{Lemma 4 (2)}}{B \wedge (A \vee C) \to A \vee (B \wedge C)}}{(A \wedge (A \vee C)) \vee (B \wedge (A \vee C)) \to A \vee (B \wedge C)}\ (*)$$

where the step $(*)$ is made with the help of $(\wedge \text{I})$, (A8) and (MP). From this and $(A \vee B) \wedge (A \vee C) \to (A \wedge (A \vee C)) \vee (B \wedge (A \vee C))$, which is derived from (A9), we obtain $(A \vee B) \wedge (A \vee C) \to A \vee (B \wedge C)$ by (Tr). □

Lemma 5. *The following rules are derivable in* $\mathbf{H}\mathcal{X}$:

$$\frac{A}{C \to D \vee A}\ (\text{R1}) \qquad \frac{A_1 \to A}{(C \to D \vee A_1) \to (C \to D \vee A)}\ (\text{R2})$$

$$\frac{A_1 \wedge A_2 \to A}{(C \to D \vee A_1) \wedge (C \to D \vee A_2) \to (C \to D \vee A)}\ (\text{R3})$$

Proof. Here we only show (R3). For this, we have the following proof.

$$\dfrac{\dfrac{\text{Lemma 4 (3)}}{(D \vee A_1) \wedge (D \vee A_2) \to D \vee (A_1 \wedge A_2)} \quad \dfrac{A_1 \wedge A_2 \to A}{D \vee (A_1 \wedge A_2) \to D \vee A}\ (*)}{(D \vee A_1) \wedge (D \vee A_2) \to D \vee A}\ (\text{Tr})$$

where the step $(*)$ is made with the help of Lemma 4 (1) and (MP). From this and $(C \to D \vee A_1) \wedge (C \to D \vee A_2) \to (C \to (D \vee A_1) \wedge (D \vee A_2))$, which is an instance of (A5), we obtain $(C \to D \vee A_1) \wedge (C \to D \vee A_2) \to (C \to D \vee A)$ by (Tr 3). □

To prove Lemma 8, we show three auxiliary lemmas.

Lemma 10. *The formula* $(D \vee (E \to A)) \wedge (A \to B) \to D \vee (E \to B)$ *is provable in* $\mathbf{H}\mathcal{X}$.

Proof. By Lemma 4 (2), we have $(D \vee (E \to A)) \wedge (A \to B) \to D \vee ((E \to A) \wedge (A \to B))$. From this and $D \vee ((E \to A) \wedge (A \to B)) \to D \vee (E \to B)$, which is derived from (A2) and Lemma 4 (1), we obtain $(D \vee (E \to A)) \wedge (A \to B) \to D \vee (E \to B)$ by (Tr). □

Lemma 11. *The following rule is derivable in* $\mathbf{H}\mathcal{X}$:

$$\frac{C \to (A \to B)}{(C \to D \vee (E \to A)) \to (C \to D \vee (E \to B))}$$

Proof. We have the following proof.

$$\dfrac{\dfrac{(\text{A1})}{(C \to D \vee (E \to A)) \to (C \to D \vee (E \to A))} \quad \dfrac{C \to (A \to B)}{(C \to D \vee (E \to A)) \to (C \to (A \to B))}\ (\text{AF})}{(C \to D \vee (E \to A)) \to (C \to D \vee (E \to A)) \wedge (C \to (A \to B))}\ (\to\wedge\text{I})$$

From this and $(C \to D \vee (E \to A)) \wedge (C \to (A \to B)) \to (C \to (D \vee (E \to A)) \wedge (A \to B))$, which is an instance of (A5), we have $(C \to D \vee (E \to A)) \to (C \to (D \vee (E \to A)) \wedge (A \to B))$ by (Tr). Then by Lemma 10 and (Tr 3), we obtain $(C \to D \vee (E \to A)) \to (C \to D \vee (E \to B))$. □

Lemma 12. *The following formulas are provable in* $\mathbf{H}\mathcal{X}$:

1. $((A \to B) \wedge C \to D \vee (E \to F \vee A))$
   $\to ((A \to B) \wedge C \to D \vee (E \to F \vee (B \wedge E)))$,

2. $(A \to F \vee (B \wedge E)) \wedge (B \wedge E \to F) \to (A \to F)$.

The following formula is provable in $\mathbf{H}\mathsf{KT}^I/\mathbf{H}\mathsf{S4}^I$:

3. $((A \to B) \wedge C \to D \vee A) \to ((A \to B) \wedge C \to D \vee (B \wedge C))$.

Proof. 1. First we consider the following two proofs.

$$\dfrac{\dfrac{\overset{\text{(A3)}}{(A \to B) \wedge C \to (A \to B)} \quad \overset{\text{Lemma 4 (1)}}{(A \to B) \to (F \vee A \to F \vee B)}}{(A \to B) \wedge C \to (F \vee A \to F \vee B)}\,\text{(Tr)}}{((A \to B) \wedge C \to D \vee (E \to F \vee A)) \to ((A \to B) \wedge C \to D \vee (E \to F \vee B))}\,\text{Lemma 11}$$

$$\dfrac{\dfrac{\overset{(*)}{(E \to F \vee B) \to (E \to E \wedge (F \vee B))} \quad \overset{\text{Lemma 4 (2)}}{E \wedge (F \vee B) \to F \vee (B \wedge E)}}{(E \to F \vee B) \to (E \to F \vee (B \wedge E))}\,\text{(Tr 3)}}{D \vee (E \to F \vee B) \to D \vee (E \to F \vee (B \wedge E))}\,\text{Lemma 4 (1), (MP)}$$

where $(*)$ is derived from (A5), (Tr), etc. From these two proofs, we obtain the required formula by (Tr 3).

2. Let $X = A \to F \vee (B \wedge E)$ and $Y = B \wedge E \to F$. Then we have the following proof.

$$\dfrac{\overset{\text{(A3)}}{X \wedge Y \to X} \quad \dfrac{\dfrac{\dfrac{\overset{\text{(A1)}}{F \to F}}{X \wedge Y \to (F \to F)}\,\text{(AF)} \quad \overset{\text{(A4)}}{X \wedge Y \to Y}}{X \wedge Y \to (F \to F) \wedge Y}\,(\to\wedge\text{I})}{X \wedge Y \to (F \vee (B \wedge E) \to F)}\,(*)}{X \wedge Y \to (A \to F)}\,(\to\text{Tr})$$

where the step $(*)$ is made with the help of (A8) and (Tr).

3. Let $Z = (A \to B) \wedge C$. Then we have the following proof.

$$\dfrac{\overset{(*)}{(Z \to D \vee A) \to (Z \to Z \wedge (D \vee A))} \quad \overset{\text{Lemma 4 (2)}}{Z \wedge (D \vee A) \to D \vee (A \wedge Z)}}{(Z \to D \vee A) \to (Z \to D \vee (A \wedge Z))}\,\text{(Tr 3)}$$

where $(*)$ is derived from (A5), (Tr), etc. Since $D \vee (A \wedge Z) \to D \vee (B \wedge C)$ is derived using (A17) etc., we obtain the required formula by (Tr 3). □

Lemma 8. *The following formulas are provable in* $\mathbf{H}\mathcal{X}$:

1. $A \wedge C \to D \vee A$,

2. $\bot \wedge C \to D$,

3. $(C \to D) \to (A \wedge C \to D)$,

4. $(C \to D) \to (C \to D \vee B)$,

5. $(C \to D \vee A) \wedge (C \to D \vee B) \to (C \to D \vee (A \wedge B))$, *6.* $(A \wedge C \to D) \wedge (B \wedge C \to D) \to ((A \vee B) \wedge C \to D)$,

7. $(C \to D \vee (E \to F \vee A)) \wedge (C \to D \vee (B \wedge E \to F))$
$\to ((A \to B) \wedge C \to D \vee (E \to F))$,

8. $(A[y/x] \wedge C \to D) \to (\forall x A \wedge C \to D)$,

9. $(C \to D \vee A[y/x]) \to (C \to D \vee \exists x A)$.

The following formula is provable in $\mathbf{H}\mathsf{KT}^I/\mathbf{H}\mathsf{S4}^I$:

10. $(C \to D \vee A) \wedge (B \wedge C \to D) \to ((A \to B) \wedge C \to D)$.

The following formula is provable in $\mathbf{H}\mathsf{K4}^I/\mathbf{H}\mathsf{S4}^I$:

11. $(C \to D \vee ((A \to B) \wedge E \to F)) \to ((A \to B) \wedge C \to D \vee (E \to F))$.

The following formula is provable in $\mathbf{H}\mathsf{BQL}^I$:

12. $(C \to D \vee (A \wedge E \to F)) \to (A \wedge C \to D \vee (E \to F))$.

Proof. Here we show 6, 7, 10 and 12. (11 is proved similarly to 12.)

6. From $(A \vee B) \wedge C \to (A \wedge C) \vee (B \wedge C)$, which is obtained from (A9), and $(A \wedge C \to D) \wedge (B \wedge C \to D) \to ((A \wedge C) \vee (B \wedge C) \to D)$, which is an instance of (A8), we derive the formula by (Tr 2).
7. Let $X = C \to D \vee (E \to F \vee A)$ and $Y = C \to D \vee (B \wedge E \to F)$. Then we have

$$\frac{\overset{\text{(A3)}}{X \wedge Y \to X} \quad \dfrac{\dfrac{\overset{\text{(A4)}}{(A \to B) \wedge C \to C}}{X \to ((A \to B) \wedge C \to D \vee (E \to F \vee A))}\,\text{(Suff)}}{X \to ((A \to B) \wedge C \to D \vee (E \to F \vee (B \wedge E)))}\,(*)}{X \wedge Y \to ((A \to B) \wedge C \to D \vee (E \to F \vee (B \wedge E)))}\,\text{(Tr)}$$

where the step $(*)$ is made with the help of Lemma 12 (1) and (Tr). On the other hand,

$$\frac{\overset{\text{(A4)}}{(A \to B) \wedge C \to C} \quad \overset{\text{(A4)}}{X \wedge Y \to Y}}{X \wedge Y \to ((A \to B) \wedge C \to D \vee (B \wedge E \to F))}\,\text{(Tr 2)}$$

Hence $X \wedge Y \to (Z \to D \vee (E \to F \vee (B \wedge E))) \wedge (Z \to D \vee (B \wedge E \to F))$ by $(\to\wedge\mathrm{I})$, where $Z = (A \to B) \wedge C$. Now,

$$\frac{\overset{\text{Lemma 12 (2)}}{(E \to F \vee (B \wedge E)) \wedge (B \wedge E \to F) \to (E \to F)}}{(Z \to D \vee (E \to F \vee (B \wedge E))) \wedge (Z \to D \vee (B \wedge E \to F)) \to (Z \to D \vee (E \to F))}\,\text{(R3)}$$

Therefore we obtain $X \wedge Y \to (Z \to D \vee (E \to F))$ by (Tr).

10. Let $X = C \to D \vee A$ and $Y = B \wedge C \to D$. Then we have

$$\dfrac{\overset{\text{(A3)}}{X \wedge Y \to X} \quad \dfrac{\dfrac{\overset{\text{(A4)}}{(A \to B) \wedge C \to C}}{X \to ((A \to B) \wedge C \to D \vee A)}\,\text{(Suff)}}{X \to ((A \to B) \wedge C \to D \vee (B \wedge C))}\,(*)}{X \wedge Y \to ((A \to B) \wedge C \to D \vee (B \wedge C))}\,\text{(Tr)}$$

where the step $(*)$ is made with the help of Lemma 12 (3) and (Tr). On the other hand, we have $X \wedge Y \to Y$ by (A4), and hence $X \wedge Y \to (Z \to D \vee (B \wedge C)) \wedge Y$ by $(\to\wedge\mathrm{I})$, where $Z = (A \to B) \wedge C$. Now, by Lemma 12 (2), we have $(Z \to D \vee (B \wedge C)) \wedge Y \to (Z \to D)$. Therefore we obtain $X \wedge Y \to (Z \to D)$ by (Tr).

12. First we note that $A \to (E \to A \wedge E)$ is provable in $\mathbf{H}\mathsf{BQL}^I$ as follows.

$$\dfrac{\dfrac{\overset{\text{(A19)}}{A \to (E \to A)} \quad \dfrac{\overset{\text{(A1)}}{E \to E}}{A \to (E \to E)}\,\text{(AF)}}{A \to (E \to A) \wedge (E \to E)}\,(\to\wedge\mathrm{I}) \quad \overset{\text{(A5)}}{(E \to A) \wedge (E \to E) \to (E \to A \wedge E)}}{A \to (E \to A \wedge E)}\,\text{(Tr)}$$

Now, let $X = C \to D \vee (A \wedge E \to F)$. Then we have the following two proofs.

$$\dfrac{\overset{\text{(A3)}}{A \wedge C \to A} \quad \dfrac{A \to (E \to A \wedge E)}{X \to (A \to (E \to A \wedge E))}\,\text{(AF)}}{X \to (A \wedge C \to (E \to A \wedge E))}\,\text{(Tr 2)}$$

$$\dfrac{\overset{\text{(A4)}}{A \wedge C \to C} \quad \overset{\text{(A1)}}{X \to (C \to D \vee (A \wedge E \to F))}}{X \to (A \wedge C \to D \vee (A \wedge E \to F))}\,\text{(Tr 2)}$$

From these, we derive $X \to (A \wedge C \to (E \to A \wedge E) \wedge (D \vee (A \wedge E \to F)))$ by $(\to\wedge\mathrm{I})$, (A5) and (Tr). On the other hand, we have $(E \to A \wedge E) \wedge (D \vee (A \wedge E \to F)) \to D \vee (E \to F)$ by a similar argument to the proof of Lemma 10. Hence we obtain $X \to (A \wedge C \to D \vee (E \to F))$ by (Tr 3). □