Simple Proofs of Characterizing Strong Normalization for Explicit Substitution Calculi

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Abstract. We present a method of lifting to explicit substitution calculi some characterizations of the strongly normalizing terms of λ-calculus by means of intersection type systems. The method is first illustrated by applying to a composition-free calculus of explicit substitutions, yielding a simpler proof than the previous one by Lengrand et al. Then we present a new intersection type system in the style of sequent calculus, and show that it characterizes the strongly normalizing terms of Dyckhoff and Urban's extension of Herbelin's explicit substitution calculus.

1 Introduction

Explicit substitution calculi were introduced for improving implementations of functional programming languages based on λ-calculus. In those calculi, substitution is not treated as a meta-operation on terms but rather as a new operator in the language. Since operational properties of substitution are studied in the object-level, unexpected behavior at the time of implementation is minimized. Also, a fine-grained control of substitution is made available; for instance, we may delay substitutions in order to avoid unnecessary duplication of information.

When augmenting λ-calculus with explicit substitutions, the evaluation process is refined by reduction rules to deal with substitutions. This suggests that reduction properties of explicit substitution calculi may vary from those of the original λ-calculus. In fact, as shown by Melliès [17], there are simply typed λ-terms that are not strongly normalizing when evaluated by the reduction rules of the explicit substitution calculus in [1].

In this paper we first study a composition-free calculus of explicit substitutions $\lambda\mathtt{x}$ [5], in which strong normalization holds for simply typed terms. In [9], Dougherty and Lescanne presented intersection type assignment systems for $\lambda\mathtt{x}$, and showed that the terms typable in one of their systems are strongly normalizing. For the original λ-calculus, the converse also holds, i.e., all strongly normalizing terms are typable in an intersection type system [19]. The system in [9], however, does not satisfy this property. Then an extended system was developed in [16] where the typable terms coincide with the strongly normalizing ones. In the first half of this paper, we give a much simpler proof of this result than the one in [16], illustrating how to lift the characterization result for the original λ-calculus to an explicit substitutions setting.

In the latter half of the paper, we apply our method to an explicit substitution calculus studied in [11, 15]. This calculus is to sequent calculus what $\lambda\mathtt{x}$-calculus is to natural deduction. Simply typed terms of the calculus correspond to proofs in Herbelin's sequent calculus [12], and the reduction rules correspond to cut-elimination steps in the sequent calculus. Although a classical variant of Herbelin's calculus was also developed in [7], we consider in this paper the calculus of [11] because it is closer to the original λ-calculus and so better for a first study of intersection type assignment systems based on sequent calculus. We introduce a new type assignment system for the explicit substitution calculus, and show that the typable terms coincide with the strongly normalizing ones. Our proof method successfully applies to the calculus, while the method in [16] seems difficult to apply.

Strong normalization proofs for typable terms in [9, 16] use a variant of the reducibility method, but they also rely on two lemmas in Section 3 of [9] whose proofs are rather complicated. On the other hand, our proof of strong normalization relies on a theorem that was proved in [4] using recursive path ordering [8] and an encoding of λ-terms with explicit substitutions into a first-order rewriting system. Thus, in our proof, the complicated part is solved by a well-established result in term rewriting. This method was used for the simply typed case of the explicit substitution calculus in [11] (see also the remark after Theorem 3).

To prove that all strongly normalizing terms are typable, we develop a novel technique. The method in [16] uses a specific perpetual strategy or an inductive characterization of strongly normalizing terms in the style of [20, 6]. However, such a perpetual strategy or an inductive characterization is difficult to spell out when considering a more complex explicit substitution calculus than $\lambda\mathtt{x}$. The idea behind our technique is instead that strongly normalizing terms are closed under $\mathtt{x}$-conversion as far as decent terms are concerned (decent terms are terms in which every substitution body is strongly normalizing). This was pointed out in [14] for $\lambda\mathtt{x}^-$ ($\lambda\mathtt{x}$ with restricted garbage collection), and in [15] for the explicit substitution calculus of [11]. For an inductive argument to work, we introduce the notion of typably decent terms, which are defined as the terms in which every substitution body is typable. It is then sufficient to show that typable terms are closed under $\mathtt{x}$-conversion as far as typably decent terms are concerned.

The paper is organized as follows. In Section 2 we introduce $\lambda\mathtt{x}$-calculus. In Section 3 we characterize the strongly normalizing $\lambda\mathtt{x}$-terms by an intersection type system in [16]. In Section 4 we introduce $\overline{\lambda}\mathtt{x}$-calculus. In Section 5 we characterize the strongly normalizing $\overline{\lambda}\mathtt{x}$-terms by a new intersection type system.

2 $\lambda\mathtt{x}$-calculus

In this section we recall the definition and some properties of $\lambda\mathtt{x}$-calculus [5, 3]. This calculus is known as the simplest explicit substitution calculus; it is up to α-conversion and uses the minimal apparatus for substitution.

The syntax and the reduction rules of $\lambda\mathtt{x}$-calculus are given in Table 1. The set of terms is denoted by $\mathcal{T}_{\lambda\mathtt{x}}$ and they are called $\lambda\mathtt{x}$-*terms*. In $M\langle x := N\rangle$,

Table 1. $\lambda\mathtt{x}$-calculus

$M, N ::= x \mid MN \mid \lambda x.M \mid M\langle x := N\rangle$
$(Beta)\quad (\lambda x.M)N \to M\langle x := N\rangle$ $(App)\quad (MM')\langle x := N\rangle \to M\langle x := N\rangle M'\langle x := N\rangle$ $(Abs)\quad (\lambda y.M)\langle x := N\rangle \to \lambda y.M\langle x := N\rangle$ $(Var)\quad x\langle x := N\rangle \to N$ $(gc)\quad M\langle x := N\rangle \to M \quad \text{if } x \notin FV(M)$

$\langle x := N\rangle$ is called an *explicit substitution* or simply substitution and N is called the *body* of the substitution. The notions of free and bound variables are defined as usual, with an additional clause that the variable x in $M\langle x := N\rangle$ binds the free occurrences of x in M. The set of free variables of a $\lambda\mathtt{x}$-term M is denoted by $FV(M)$. The symbol $\equiv$ denotes syntactical equality modulo α-conversion.

The notion of $\lambda\mathtt{x}$-reduction is defined by the contextual closures of all reduction rules in Table 1. We use $\to_{\lambda\mathtt{x}}$ for one-step reduction, $\xrightarrow{+}_{\lambda\mathtt{x}}$ for its transitive closure, and $\xrightarrow{*}_{\lambda\mathtt{x}}$ for its reflexive transitive closure. The set of $\lambda\mathtt{x}$-terms that are strongly normalizing with respect to $\lambda\mathtt{x}$-reduction is denoted by $\mathcal{SN}_{\lambda\mathtt{x}}$. These kinds of notations are also used for the notions of other reductions introduced in this paper.

The subcalculus of $\lambda\mathtt{x}$ without the rule $(Beta)$ is denoted by $\mathtt{x}$. This subcalculus has the following properties [3].

Proposition 1. *The subcalculus* $\mathtt{x}$ *is strongly normalizing and confluent.*

Proof. Strong normalization is shown by defining a map $h : \mathcal{T}_{\lambda\mathtt{x}} \longrightarrow \mathbb{N}$ which decreases on $\mathtt{x}$-reduction; define

$$\begin{array}{llll} h(x) & =_{def}\ 1 & h(MN) & =_{def}\ h(M) + h(N) + 1 \\ h(\lambda x.N) & =_{def}\ h(N) + 1 & h(M\langle x := N\rangle) & =_{def}\ h(M) \times (h(N) + 1) \end{array}$$

and observe that if $M \to_{\mathtt{x}} N$ then $h(M) > h(N)$. To prove confluence, it is now sufficient to show local confluence, which is easy. □

As a result, we can define the unique $\mathtt{x}$-normal form of each $\lambda\mathtt{x}$-term.

Definition 1. *The unique* $\mathtt{x}$*-normal form of a* $\lambda\mathtt{x}$*-term* M *is denoted by* $\mathtt{x}(M)$*.*

The usual λ-terms are the $\lambda\mathtt{x}$-terms that do not contain explicit substitutions. In this paper they are called *pure λ-terms.* The β-rule on pure λ-terms is stated as $(\lambda x.M)N \to_\beta M[N/x]$ where $M[N/x]$ represents meta-substitution. The relation between pure λ-terms and $\mathtt{x}$-normal forms is as follows.

Proposition 2. *M is a pure λ-term if and only if M is in* $\mathtt{x}$*-normal form.*

Proof. The left-to-right implication is straightforward. We prove the converse by induction on the structure of M. Suppose that M is in $\mathtt{x}$-normal form. Then by the induction hypothesis, all subterms of M are pure λ-terms. Now, if M is not a pure λ-term, then M is of the form $P\langle x := Q\rangle$ where P, Q are pure λ-terms. In this case, M is an $\mathtt{x}$-redex, which is a contradiction. □

The next proposition shows that the subcalculus $\mathtt{x}$ correctly simulates meta-substitution on pure λ-terms.

Proposition 3. *Let M, N be pure λ-terms. Then $M\langle x := N\rangle \xrightarrow{*}_{\mathtt{x}} M[N/x]$.*

Proof. By induction on the structure of M. □

The next lemma shows that $\lambda\mathtt{x}$-reduction simulates β-reduction.

Lemma 1. *Let M, N be pure λ-terms. If $M \to_\beta N$ then $M \xrightarrow{+}_{\lambda\mathtt{x}} N$.*

Proof. By induction on the reduction relation $\to_\beta$. We consider the case where $M \equiv (\lambda x.P)Q \to_\beta P[Q/x] \equiv N$. Then use $\to_{Beta}$ to create $P\langle x := Q\rangle$, and use Proposition 3 to reach $P[Q/x]$. □

Bloo and Geuvers [4] proved the following theorem to show that $\lambda\mathtt{x}$-calculus satisfies the PSN property. Their method appeals to recursive path ordering [8] and a first-order encoding of $\lambda\mathtt{x}$-terms. We use the theorem to prove one direction in characterizing strongly normalizing $\lambda\mathtt{x}$-terms by means of intersection types.

Definition 2 (Bounded terms). *The set of* bounded terms, *denoted $\lambda\mathtt{x}^{<\infty}$, is defined by $\lambda\mathtt{x}^{<\infty} =_{def} \{M \mid \text{for every subterm } N \text{ of } M,\ \mathtt{x}(N) \in \mathcal{SN}_\beta\}$.*

Theorem 1 ([4])**.** *If $M \in \lambda\mathtt{x}^{<\infty}$ then $M \in \mathcal{SN}_{\lambda\mathtt{x}}$.*

3 Characterization of Strongly Normalizing λx-terms

In this section we show that the strongly normalizing $\lambda\mathtt{x}$-terms are characterized by typability in an intersection type assignment system given in [16]. To prove that the typable terms are strongly normalizing, we use Theorem 1, subject reduction (Theorem 2), and the result in [19] for ordinary λ-calculus and intersection types. (Some of the proofs of lemmas are taken from [16].) To prove the other direction, we make an inductive argument together with the preservation of types under a certain $\mathtt{x}$-expansion.

First, the set of types is defined by the grammar: $\sigma ::= \varphi \mid \sigma \to \sigma \mid \sigma \cap \sigma$ where φ ranges over a denumerable set of type atoms. We use letters $\sigma, \tau, \rho, \ldots$ for arbitrary types. The type assignment system $\lambda\mathtt{x}_\cap$ is defined by the rules in Table 2. A *typing context* is defined as a finite set of pairs $\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$ where the variables are pairwise distinct. The typing context $\Gamma, x : \sigma$ denotes the union $\Gamma \cup \{x : \sigma\}$ where $x \notin \Gamma$ ($x \notin \Gamma$ means that x does not appear in Γ).

Table 2. The type assignment system $\lambda\mathtt{x}_\cap$

$$\frac{}{\Gamma, x:\sigma \vdash x:\sigma}\ (Ax) \qquad \frac{\Gamma \vdash M:\sigma\to\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}\ (\to E)$$

$$\frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma\to\tau}\ (\to I) \qquad \frac{\Gamma \vdash M:\sigma \quad \Gamma \vdash M:\tau}{\Gamma \vdash M:\sigma\cap\tau}\ (\cap I) \qquad \frac{\Gamma \vdash M:\sigma_1\cap\sigma_2}{\Gamma \vdash M:\sigma_i}\ (\cap E)\ \text{where } i\in\{1,2\}$$

$$\frac{\Gamma \vdash N:\sigma \quad \Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash M\langle x:=N\rangle:\tau}\ (Cut) \qquad \frac{\Delta \vdash N:\sigma \quad \Gamma \vdash M:\tau}{\Gamma \vdash M\langle x:=N\rangle:\tau}\ (K\text{-}cut)\ \text{where } x\notin\Gamma$$

We write $\Gamma \vdash M : \sigma$ if there exists a derivation in $\lambda\mathtt{x}_\cap$ that has this judgement as its conclusion.

The original intersection type assignment system for $\lambda\mathtt{x}$ in [9] does not have the rule (K-cut) and is not enough to type all strongly normalizing terms. For example, the $\lambda\mathtt{x}$-term $z\langle y := xx\rangle\langle x := \lambda a.aa\rangle$ is strongly normalizing but not typable in the system of [9]. For more discussions, see [16, p. 29].

The system $\lambda\mathtt{x}_\cap$ has some unusual features caused by the rule (K-cut). For instance, x does not necessarily appear in Γ of $\Gamma \vdash M : \tau$ even if $x \in FV(M)$. Nevertheless, we can appropriately rename variables with careful treatment.

Lemma 2.

1. *If $\Gamma \vdash M : \tau$, $y \notin \Gamma$ and $y \notin FV(M)$ then $\Gamma[y/x] \vdash M[y/x] : \tau$.*
2. *If $\Gamma \vdash M : \tau$ and $x \notin \Gamma$ then $\Gamma, x : \sigma \vdash M : \tau$.*
3. *If $\Gamma, x : \sigma \vdash M : \tau$ and $x \notin FV(M)$ then $\Gamma \vdash M : \tau$.*

Proof. By induction on the structure of derivations. □

Since terms typed in an intersection type system do not in general reflect the structure of their typing derivations, a Generation Lemma is necessary for proving the subject reduction and expansion properties. Below we give a precise statement of Generation Lemma for the case of $\lambda\mathtt{x}_\cap$. To do so we first define a pre-ordering on types.

Definition 3. *The relation $\leqslant$ on types is defined by the following axioms and rules:*

1. $\sigma \leqslant \sigma$
2. $\sigma\cap\tau \leqslant \sigma,\ \sigma\cap\tau \leqslant \tau$
3. $\sigma \leqslant \tau,\ \tau \leqslant \rho \Rightarrow \sigma \leqslant \rho$
4. $\sigma \leqslant \tau,\ \sigma \leqslant \rho \Rightarrow \sigma \leqslant \tau\cap\rho$

Lemma 3. *If $\Gamma \vdash M : \sigma$ and $\sigma \leqslant \tau$ then $\Gamma \vdash M : \tau$.*

Proof. By induction on the definition of $\leqslant$. □

Lemma 4. *If $\Gamma, x : \sigma \vdash M : \tau$ and $\rho \leqslant \sigma$ then $\Gamma, x : \rho \vdash M : \tau$.*

Proof. By induction on the derivation of $\Gamma, x : \sigma \vdash M : \tau$. □

In the following, we use $\underline{n}$ for $\{1, \ldots, n\}$, and $\cap_{\underline{n}}\sigma_i$ for $\sigma_1 \cap \cdots \cap \sigma_n$.

Lemma 5. *Let $\cap_{\underline{m}}\sigma_i \leqslant \cap_{\underline{n}}\tau_j$ where none of the σ_i ($i \in \underline{m}$) and τ_j ($j \in \underline{n}$) is an intersection. Then, for each τ_j, there exists σ_i such that $\sigma_i = \tau_j$.*

Proof. By induction on the definition of $\leqslant$. □

Now we state a precise form of Generation Lemma for the system $\lambda\mathtt{x}_\cap$.

Lemma 6 (Generation Lemma).

1. *$\Gamma \vdash x : \tau$ if and only if there exists $x : \sigma \in \Gamma$ such that $\sigma \leqslant \tau$.*
2. *$\Gamma \vdash MN : \tau$ if and only if there exist $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n$ ($n \geqslant 1$) such that $\cap_{\underline{n}}\tau_i \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma \vdash M : \sigma_i \to \tau_i$ and $\Gamma \vdash N : \sigma_i$.*
3. *$\Gamma \vdash \lambda x.M : \tau$ if and only if there exist $\sigma_1, \ldots, \sigma_n, \rho_1, \ldots, \rho_n$ ($n \geqslant 1$) such that $\cap_{\underline{n}}(\sigma_i \to \rho_i) \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma, x : \sigma_i \vdash M : \rho_i$.*
4. *$\Gamma \vdash \lambda x.M : \sigma \to \tau$ if and only if $\Gamma, x : \sigma \vdash M : \tau$.*
5. *$\Gamma \vdash M\langle x := N\rangle : \tau$ if and only if either*
   (a) there exists σ such that $\Gamma \vdash N : \sigma$ and $\Gamma, x : \sigma \vdash M : \tau$, or
   (b) $\Gamma \vdash M : \tau$ ($x \notin \Gamma$) and there exist Δ, σ such that $\Delta \vdash N : \sigma$.

Proof. The right-to-left implications immediately follow from the typing rules and Lemma 3. The converses are proved by induction on the structure of derivations, except for part 4 which follows from part 3 and Lemma 5. Here we consider the case in part 5 where the last applied rule in the derivation is $(\cap I)$:

$$\frac{\Gamma \vdash M\langle x := N\rangle : \tau_1 \quad \Gamma \vdash M\langle x := N\rangle : \tau_2}{\Gamma \vdash M\langle x := N\rangle : \tau_1 \cap \tau_2}\ (\cap I)$$

In this case, by the induction hypothesis, we have the following four possibilities:

(i) there exist σ_1, σ_2 such that $\Gamma \vdash N : \sigma_1$, $\Gamma, x : \sigma_1 \vdash M : \tau_1$, $\Gamma \vdash N : \sigma_2$ and $\Gamma, x : \sigma_2 \vdash M : \tau_2$. In this case, by Lemma 4, $\Gamma, x : \sigma_1 \cap \sigma_2 \vdash M : \tau_1$ and $\Gamma, x : \sigma_1 \cap \sigma_2 \vdash M : \tau_2$, so by the rule $(\cap I)$, $\Gamma, x : \sigma_1 \cap \sigma_2 \vdash M : \tau_1 \cap \tau_2$. On the other hand, by the rule $(\cap I)$, we have $\Gamma \vdash N : \sigma_1 \cap \sigma_2$. Hence (a) holds for $\sigma = \sigma_1 \cap \sigma_2$.

(ii) there exist $\sigma_1, \Delta, \sigma_2$ such that $\Gamma \vdash N : \sigma_1$, $\Gamma, x : \sigma_1 \vdash M : \tau_1$, $\Gamma \vdash M : \tau_2$ and $\Delta \vdash N : \sigma_2$. In this case, by Lemma 2 (2), we have $\Gamma, x : \sigma_1 \vdash M : \tau_2$, so by the rule $(\cap I)$, $\Gamma, x : \sigma_1 \vdash M : \tau_1 \cap \tau_2$. Hence (a) holds for $\sigma = \sigma_1$.

(iii) there exist $\Delta, \sigma_1, \sigma_2$ such that $\Gamma \vdash M : \tau_1$, $\Delta \vdash N : \sigma_1$, $\Gamma \vdash N : \sigma_2$ and $\Gamma, x : \sigma_2 \vdash M : \tau_2$. This case is proved similarly to the case (ii).

(iv) there exist $\Delta, \sigma_1, \Delta', \sigma_2$ such that $\Gamma \vdash M : \tau_1$ ($x \notin \Gamma$), $\Delta \vdash N : \sigma_1$, $\Gamma \vdash M : \tau_2$ and $\Delta' \vdash N : \sigma_2$. In this case, by the rule $(\cap I)$, we have $\Gamma \vdash M : \tau_1 \cap \tau_2$. Hence (b) holds. □

The next lemma shows that the root reduction and expansion by the rules (App) and (Abs) preserve types.

Lemma 7.

1. $\Gamma \vdash (MM')\langle x := N\rangle : \tau$ *if and only if* $\Gamma \vdash M\langle x := N\rangle M'\langle x := N\rangle : \tau$.
2. $\Gamma \vdash (\lambda y.M)\langle x := N\rangle : \tau$ *if and only if* $\Gamma \vdash \lambda y.M\langle x := N\rangle : \tau$ $(y \notin FV(N))$.

Proof. Here we only show part 1. (For part 2, see Appendix A.)

($\Rightarrow$) Let $\Gamma \vdash (MM')\langle x := N\rangle : \tau$. Then by Lemma 6 (5), we have two cases:

(i) there exists σ such that $\Gamma \vdash N : \sigma$ and $\Gamma, x : \sigma \vdash MM' : \tau$. In this case, by Lemma 6 (2), there exist $\rho_1, \ldots, \rho_n, \tau_1, \ldots, \tau_n$ such that $\cap_{\underline{n}}\tau_i \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma, x : \sigma \vdash M : \rho_i \to \tau_i$ and $\Gamma, x : \sigma \vdash M' : \rho_i$. Then by the rule (Cut), for each $i \in \underline{n}$, $\Gamma \vdash M\langle x := N\rangle : \rho_i \to \tau_i$ and $\Gamma \vdash M'\langle x := N\rangle : \rho_i$, so by the rule $(\to E)$, $\Gamma \vdash M\langle x := N\rangle M'\langle x := N\rangle : \tau_i$. Hence by the rule $(\cap I)$, $\Gamma \vdash M\langle x := N\rangle M'\langle x := N\rangle : \cap_{\underline{n}}\tau_i$, and by Lemma 3, we obtain $\Gamma \vdash M\langle x := N\rangle M'\langle x := N\rangle : \tau$.

(ii) $\Gamma \vdash MM' : \tau$ $(x \notin \Gamma)$ and there exist Δ, σ such that $\Delta \vdash N : \sigma$. This case is proved similarly to the case (i), using $(K\text{-}cut)$ instead of (Cut).

($\Leftarrow$) Let $\Gamma \vdash M\langle x := N\rangle M'\langle x := N\rangle : \tau$. Then by Lemma 6 (2), there exist $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n$ such that $\cap_{\underline{n}}\tau_i \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma \vdash M\langle x := N\rangle : \sigma_i \to \tau_i$ and $\Gamma \vdash M'\langle x := N\rangle : \sigma_i$. By Lemma 6 (5), for each $i \in \underline{n}$, there are four possibilities:

(i) there exist ρ, ν such that $\Gamma \vdash N : \rho$, $\Gamma, x : \rho \vdash M : \sigma_i \to \tau_i$, $\Gamma \vdash N : \nu$ and $\Gamma, x : \nu \vdash M' : \sigma_i$. In this case, by Lemma 4, $\Gamma, x : \rho \cap \nu \vdash M : \sigma_i \to \tau_i$ and $\Gamma, x : \rho \cap \nu \vdash M' : \sigma_i$, so by the rule $(\to E)$, $\Gamma, x : \rho \cap \nu \vdash MM' : \tau_i$. On the other hand, by the rule $(\cap I)$, we have $\Gamma \vdash N : \rho \cap \nu$. Hence by the rule (Cut), we get $\Gamma \vdash (MM')\langle x := N\rangle : \tau_i$.

(ii) there exist ρ, Δ, ν such that $\Gamma \vdash N : \rho$, $\Gamma, x : \rho \vdash M : \sigma_i \to \tau_i$, $\Gamma \vdash M' : \sigma_i$ and $\Delta \vdash N : \nu$. In this case, by Lemma 2 (2), we have $\Gamma, x : \rho \vdash M' : \sigma_i$, so by the rule $(\to E)$, $\Gamma, x : \rho \vdash MM' : \tau_i$. Hence by the rule (Cut), we get $\Gamma \vdash (MM')\langle x := N\rangle : \tau_i$.

(iii) there exist Δ, ν, ρ such that $\Gamma \vdash M : \sigma_i \to \tau_i$, $\Delta \vdash N : \nu$, $\Gamma \vdash N : \rho$ and $\Gamma, x : \rho \vdash M' : \sigma_i$. In this case, by Lemma 2 (2), we have $\Gamma, x : \rho \vdash M : \sigma_i \to \tau_i$, so by the rule $(\to E)$, $\Gamma, x : \rho \vdash MM' : \tau_i$. Hence by the rule (Cut), we get $\Gamma \vdash (MM')\langle x := N\rangle : \tau_i$.

(iv) there exist $\Delta, \nu, \Delta', \rho$ such that $\Gamma \vdash M : \sigma_i \to \tau_i$ $(x \notin \Gamma)$, $\Delta \vdash N : \nu$, $\Gamma \vdash M' : \sigma_i$ and $\Delta' \vdash N : \rho$. In this case, by the rule $(\to E)$, we have $\Gamma \vdash MM' : \tau_i$. Hence by the rule $(K\text{-}cut)$, we get $\Gamma \vdash (MM')\langle x := N\rangle : \tau_i$.

Hence by the rule $(\cap I)$, we have $\Gamma \vdash (MM')\langle x := N\rangle : \cap_{\underline{n}}\tau_i$, and by Lemma 3, we obtain $\Gamma \vdash (MM')\langle x := N\rangle : \tau$. □

Now we are in a position to show that the system $\lambda\mathtt{x}_\cap$ satisfies the subject reduction property.

Theorem 2 (Subject Reduction). *If* $M \to_{\lambda\mathtt{x}} N$ *and* $\Gamma \vdash M : \tau$ *then* $\Gamma \vdash N : \tau$.

Proof. By induction on the reduction relation $\to_{\lambda\mathtt{x}}$. First we consider the cases where the reduction is at the root.

(*Beta*): Let $\Gamma \vdash (\lambda x.P)Q : \tau$. Then by Lemma 6 (2), there exist $\sigma_1, \ldots, \sigma_n$, $\tau_1, \ldots, \tau_n$ such that $\cap_{\underline{n}}\tau_i \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma \vdash \lambda x.P : \sigma_i \to \tau_i$ and $\Gamma \vdash Q : \sigma_i$. By Lemma 6 (4), for each $i \in \underline{n}$, $\Gamma, x : \sigma_i \vdash P : \tau_i$, and so $\Gamma \vdash P\langle x := Q\rangle : \tau_i$ by the rule (*Cut*). Hence by the rule $(\cap I)$ and Lemma 3, we obtain $\Gamma \vdash P\langle x := Q\rangle : \tau$.

(*App*): Let $\Gamma \vdash (PQ)\langle x := R\rangle : \tau$. Then by Lemma 7 (1), we have $\Gamma \vdash P\langle x := R\rangle Q\langle x := R\rangle : \tau$.

(*Abs*): Let $\Gamma \vdash (\lambda y.P)\langle x := Q\rangle : \tau$. Then by Lemma 7 (2), we have $\Gamma \vdash \lambda y.P\langle x := Q\rangle : \tau$.

(*Var*): Let $\Gamma \vdash x\langle x := Q\rangle : \tau$. Then by Lemma 6 (5), either (a) there exists σ such that $\Gamma \vdash Q : \sigma$ and $\Gamma, x : \sigma \vdash x : \tau$, or (b) $\Gamma \vdash x : \tau$ ($x \notin \Gamma$) and there exist Δ, σ such that $\Delta \vdash Q : \sigma$. However, by Lemma 6 (1), (b) is impossible and we have $\sigma \leqslant \tau$ from $\Gamma, x : \sigma \vdash x : \tau$ in (a). Hence by Lemma 3, we obtain $\Gamma \vdash Q : \tau$.

(*gc*): Let $\Gamma \vdash P\langle x := Q\rangle : \tau$ and $x \notin FV(P)$. Then by Lemma 6 (5), either (a) there exists σ such that $\Gamma \vdash Q : \sigma$ and $\Gamma, x : \sigma \vdash P : \tau$, or (b) $\Gamma \vdash P : \tau$ ($x \notin \Gamma$) and there exist Δ, σ such that $\Delta \vdash Q : \sigma$. In the case (a), we have $\Gamma \vdash P : \tau$ by Lemma 2 (3).

The cases where the reduction is not at the root are immediate by Lemma 6 and the induction hypothesis. □

Now we can prove one direction of the characterization theorem of strongly normalizing $\lambda\mathtt{x}$-terms.

Theorem 3. *If M is typable in the system $\lambda\mathtt{x}_\cap$ then $M \in \mathcal{SN}_{\lambda\mathtt{x}}$.*

Proof. By Theorem 1, it suffices to show that if M is typable in the system $\lambda\mathtt{x}_\cap$ then $M \in \lambda\mathtt{x}^{<\infty}$. Let M be typable in $\lambda\mathtt{x}_\cap$ and N be any subterm of M. Then N is also typable in $\lambda\mathtt{x}_\cap$, and by Theorem 2, so is $\mathtt{x}(N)$. Since $\mathtt{x}(N)$ is a pure λ-term (Proposition 2), it is typable without using (*Cut*) and (*K-cut*). Hence by the result in [19], we have $\mathtt{x}(N) \in \mathcal{SN}_\beta$. Thus we obtain $M \in \lambda\mathtt{x}^{<\infty}$. □

In [18], a method for deriving strong normalization of typable terms from the PSN property was formalized, but it does not work for the system $\lambda\mathtt{x}_\cap$. The method requires preservation of typability during the lift of the explicit substitutions into β-redexes. However, the $\lambda\mathtt{x}$-term $z\langle y := xx\rangle\langle x := \lambda a.aa\rangle$ (the counter example for the system [9]) is typable in $\lambda\mathtt{x}_\cap$ while the result of lifting $(\lambda x.(\lambda y.z)(xx))(\lambda a.aa)$ is not, so that one cannot infer strong normalization of $z\langle y := xx\rangle\langle x := \lambda a.aa\rangle$ by that method.

Next we prove the converse of Theorem 3. For this we introduce the notion of typably decent terms.

Definition 4 (Typably decent terms). *A $\lambda\mathtt{x}$-term M is said to be* typably decent *if for every substitution $\langle x := N\rangle$ occurring in M, N is typable in $\lambda\mathtt{x}_\cap$.*

Lemma 8. *If M is typably decent, $M \to_{\mathtt{x}} N$ and $\Gamma \vdash N : \tau$, then $\Gamma \vdash M : \tau$.*

Proof. By induction on the reduction relation $\rightarrow_{\mathtt{x}}$. First we consider the cases where the reduction is at the root.

- (App): Let $\Gamma \vdash P\langle x := R\rangle Q\langle x := R\rangle : \tau$. Then by Lemma 7 (1), we have $\Gamma \vdash (PQ)\langle x := R\rangle : \tau$.
- (Abs): Let $\Gamma \vdash \lambda y.P\langle x := Q\rangle : \tau$. Then by Lemma 7 (2), we have $\Gamma \vdash (\lambda y.P)\langle x := Q\rangle : \tau$.
- (Var): Let $\Gamma \vdash Q : \tau$. Since $M \equiv x\langle x := Q\rangle$ for a fresh variable x, we have $\Gamma, x : \tau \vdash x : \tau$, and so $\Gamma \vdash x\langle x := Q\rangle : \tau$ by the rule (Cut).
- (gc): Let $\Gamma \vdash P : \tau$. Since $M \equiv P\langle x := Q\rangle$ for a fresh variable x and M is typably decent, there exist Δ, σ such that $\Delta \vdash Q : \sigma$. Hence we obtain $\Gamma \vdash P\langle x := Q\rangle : \tau$ by the rule $(K\text{-}cut)$.

The cases where the reduction is not at the root are immediate by Lemma 6 and the induction hypothesis. $\square$

Lemma 9. *If M is typably decent and $M \rightarrow_{\mathtt{x}} N$, then N is typably decent.*

Proof. By induction on the reduction relation $\rightarrow_{\mathtt{x}}$. The cases where the reduction is at the root are straightforward, since every substitution body occurring in N also occurs in M. Let us consider the case $M \equiv P\langle x := Q\rangle$ and $Q \rightarrow_{\mathtt{x}} Q'$. Since M is typably decent, Q is typable in $\lambda\mathtt{x}_\cap$, so by Theorem 2, Q' is typable in $\lambda\mathtt{x}_\cap$. Hence we see that $P\langle x := Q'\rangle$ is typably decent. $\square$

Lemma 10. *If M is typably decent, $M \xrightarrow{*}_{\mathtt{x}} N$ and $\Gamma \vdash N : \tau$, then $\Gamma \vdash M : \tau$.*

Proof. By induction on the length of the reduction steps of $M \xrightarrow{*}_{\mathtt{x}} N$, using Lemmas 8 and 9. $\square$

Now we can prove the converse of Theorem 3.

Theorem 4. *If $M \in \mathcal{SN}_{\lambda\mathtt{x}}$ then M is typable in the system $\lambda\mathtt{x}_\cap$.*

Proof. By induction on the structure of M. Suppose that $M \in \mathcal{SN}_{\lambda\mathtt{x}}$. Then for every substitution $\langle x := N\rangle$ occurring in M, $N \in \mathcal{SN}_{\lambda\mathtt{x}}$, so by the induction hypothesis, N is typable in $\lambda\mathtt{x}_\cap$. Hence M is typably decent. On the other hand, since $M \in \mathcal{SN}_{\lambda\mathtt{x}}$, we have $\mathtt{x}(M) \in \mathcal{SN}_{\lambda\mathtt{x}}$, so by Lemma 1, $\mathtt{x}(M) \in \mathcal{SN}_\beta$. Hence by the result in [19], $\mathtt{x}(M)$ is typable in $\lambda\mathtt{x}_\cap$ (without using (Cut) and $(K\text{-}cut)$). Therefore by Lemma 10, M is typable in $\lambda\mathtt{x}_\cap$. $\square$

4 $\overline{\lambda}$x-calculus

In the remainder of the paper we study an explicit substitution calculus which we call here $\overline{\lambda}$x-calculus. This calculus is to sequent calculus what λx-calculus is to natural deduction. Simply typed terms of the calculus correspond to proofs in Herbelin's sequent calculus [12], which has a stronger connection with λ-calculus than the usual sequent calculus does; in particular, it relates a unique cut-free proof to each normal term of the simply typed λ-calculus. The reduction rules of Herbelin's original calculus were later extended by Dyckhoff and Urban [11]

Table 3. $\overline{\lambda}\mathtt{x}$-calculus

$$t, u, v ::= xl \mid \lambda x.t \mid tl \mid t\langle x := v\rangle$$
$$l, l' ::= [] \mid t :: l \mid ll' \mid l\langle x := v\rangle$$

$(Beta)$	$(\lambda x.t)(u :: l) \to t\langle x := u\rangle l$
$(1a)$	$[]l \to l$
$(1b)$	$(u :: l)l' \to u :: (ll')$
$(2a)$	$[]\langle x := v\rangle \to []$
$(2b)$	$(u :: l)\langle x := v\rangle \to u\langle x := v\rangle :: l\langle x := v\rangle$
$(3a)$	$(xl)l' \to x(ll')$
$(3b)$	$(\lambda y.t)[] \to \lambda y.t$
$(4a)$	$(yl)\langle x := v\rangle \to yl\langle x := v\rangle \quad \text{if } y \not\equiv x$
$(4b)$	$(xl)\langle x := v\rangle \to vl\langle x := v\rangle$
$(4c)$	$(\lambda y.t)\langle x := v\rangle \to \lambda y.t\langle x := v\rangle$
$(5a)$	$(ll')l'' \to l(l'l'')$
$(5b)$	$(ll')\langle x := v\rangle \to l\langle x := v\rangle l'\langle x := v\rangle$
$(5c)$	$(tl)l' \to t(ll')$
$(5d)$	$(tl)\langle x := v\rangle \to t\langle x := v\rangle l\langle x := v\rangle$

to simulate full β-reduction. In the simply typed case, they correspond to cut-elimination steps in the sequent calculus.

Table 3 gives the syntax and the reduction rules of $\overline{\lambda}\mathtt{x}$-calculus. The syntax has two kinds of expressions: terms and lists of terms, ranged over by t, u, v and by l, l', respectively. The set of terms is denoted by $\mathcal{T}_{\overline{\lambda}\mathtt{x}}$ and the set of lists of terms by $\mathcal{L}_{\overline{\lambda}\mathtt{x}}$. Elements of $\mathcal{T}_{\overline{\lambda}\mathtt{x}} \cup \mathcal{L}_{\overline{\lambda}\mathtt{x}}$ are called $\overline{\lambda}\mathtt{x}$-*terms* and ranged over by a, b. The notions of free and bound variables are defined as in the case of $\lambda\mathtt{x}$-terms.

To see the relation to ordinary λ-calculus, it is useful to consider a subset of $\overline{\lambda}\mathtt{x}$-terms defined by the following grammar:

$$t, u, v ::= xl \mid \lambda x.t \mid (\lambda x.t)(u :: l)$$
$$l, l' ::= [] \mid t :: l$$

The $\overline{\lambda}\mathtt{x}$-terms generated by this grammar are called *pure terms*. Then compare the grammar of pure terms with the following inductive characterization of the set of pure λ-terms:

$$M, N ::= xM_1 \ldots M_n \mid \lambda x.M \mid (\lambda x.M)NM_1 \ldots M_n \quad (n \geqslant 0)$$

Note that this certainly generates all pure λ-terms. Now it is easy to see that there exists one-to-one correspondence between pure λ-terms and pure terms in

$\mathcal{T}_{\overline{\lambda}\mathtt{x}}$. We denote the bijection from pure λ-terms to pure terms by Ψ. Moreover we define β-reduction on pure terms in $\mathcal{T}_{\overline{\lambda}\mathtt{x}}$ so that it coincides with β-reduction on pure λ-terms under the bijection, i.e., for any pure λ-terms M, M', $M \to_\beta M'$ if and only if $\Psi(M) \to_\beta \Psi(M')$. (The β-reduction extends to pure terms in $\mathcal{L}_{\overline{\lambda}\mathtt{x}}$.)

The notion of $\overline{\lambda}\mathtt{x}$-reduction is defined by the contextual closures of all reduction rules in Table 3. Then $\overline{\lambda}\mathtt{x}$-calculus works as an explicit substitution calculus for the isomorphic image of λ-calculus. The reduction properties of $\overline{\lambda}\mathtt{x}$-calculus are similar to those of $\lambda\mathtt{x}$-calculus. In the following we summarize results on the subcalculus $\mathtt{x}$ (i.e., the calculus without the rule (*Beta*)) and the relation between $\overline{\lambda}\mathtt{x}$-reduction and β-reduction on pure terms. (For details, see [11, 15].)

Proposition 4. *The subcalculus* $\mathtt{x}$ *is strongly normalizing and confluent.*

Definition 5. *The unique* $\mathtt{x}$*-normal form of a* $\overline{\lambda}\mathtt{x}$*-term* a *is denoted by* $\mathtt{x}(a)$.

Proposition 5. a *is a pure term if and only if* a *is in* $\mathtt{x}$*-normal form.*

Lemma 11. *For any pure terms* $a, b \in \mathcal{T}_{\overline{\lambda}\mathtt{x}} \cup \mathcal{L}_{\overline{\lambda}\mathtt{x}}$, *if* $a \to_\beta b$ *then* $a \xrightarrow{+}_{\overline{\lambda}\mathtt{x}} b$.

Using a similar technique to the one in [4], Dyckhoff and Urban [11] proved the following theorem. We use this theorem to characterize strongly normalizing $\overline{\lambda}\mathtt{x}$-terms by an intersection type assignment system in the next section.

Definition 6 (Bounded terms). *The set of* bounded terms, *denoted* $\overline{\lambda}\mathtt{x}^{<\infty}$, *is defined by* $\overline{\lambda}\mathtt{x}^{<\infty} =_{def} \{a \mid \text{for every subterm } b \text{ of } a,\ \mathtt{x}(b) \in \mathcal{SN}_\beta\}$.

Theorem 5 ([11])**.** *If* $a \in \overline{\lambda}\mathtt{x}^{<\infty}$ *then* $a \in \mathcal{SN}_{\overline{\lambda}\mathtt{x}}$.

5 Characterization of Strongly Normalizing $\overline{\lambda}\mathtt{x}$-terms

In this section we introduce a new intersection type assignment system in the style of sequent calculus. The system is an extension of Herbelin's type assignment system with simple types in [12]. We show that the strongly normalizing $\overline{\lambda}\mathtt{x}$-terms coincide with those typable in the intersection type assignment system in a similar way to that in Section 3.

First we extend the pre-ordering $\leqslant$ to one in the style of [2], which reduces the difficulty of proving the subject reduction property of our intersection type assignment system.

Definition 7. *The relation* $\leqslant$ *on types is defined by the axioms and rules 1–4 in Definition 3 and the following:*

5. $(\sigma \to \tau) \cap (\sigma \to \rho) \leqslant \sigma \to (\tau \cap \rho)$ 6. $\sigma' \leqslant \sigma,\ \tau \leqslant \tau' \Rightarrow \sigma \to \tau \leqslant \sigma' \to \tau'$

Lemma 12. $(\sigma_1 \to \tau_1) \cap (\sigma_2 \to \tau_2) \leqslant (\sigma_1 \cap \sigma_2) \to (\tau_1 \cap \tau_2)$.

Lemma 13. *If* $\cap_{\underline{n}}(\mu_i \to \nu_i) \leqslant \sigma \to \tau$ *then there exist* $i_1, \ldots, i_k \in \underline{n}$ *such that* $\sigma \leqslant \mu_{i_1} \cap \cdots \cap \mu_{i_k}$ *and* $\nu_{i_1} \cap \cdots \cap \nu_{i_k} \leqslant \tau$.

Table 4. The type assignment system $\overline{\lambda}\mathbf{x}_\cap$

$$\frac{}{\Gamma;\sigma\vdash[\,]:\sigma}\ (Ax)\qquad \frac{\Gamma,x:\sigma;\sigma\vdash l:\tau}{\Gamma,x:\sigma;-\vdash xl:\tau}\ (Der)\qquad \frac{\Gamma,x:\sigma;-\vdash t:\tau}{\Gamma;-\vdash \lambda x.t:\sigma\to\tau}\ (R\to)$$

$$\frac{\Gamma;-\vdash t:\sigma\quad \Gamma;\tau\vdash l:\rho}{\Gamma;\sigma\to\tau\vdash t::l:\rho}\ (L\to)\qquad \frac{\Gamma;\Pi\vdash a:\sigma\quad \Gamma;\Pi\vdash a:\tau}{\Gamma;\Pi\vdash a:\sigma\cap\tau}\ (R\cap)$$

$$\frac{\Gamma;\sigma\vdash l:\tau\quad \rho\leqslant\sigma}{\Gamma;\rho\vdash l:\tau}\ (L\leqslant)\qquad \frac{\Gamma;\Pi\vdash a:\sigma\quad \sigma\leqslant\tau}{\Gamma;\Pi\vdash a:\tau}\ (R\leqslant)$$

$$\frac{\Gamma;\Pi\vdash a:\sigma\quad \Gamma;\sigma\vdash l:\tau}{\Gamma;\Pi\vdash al:\tau}\ (Cut_1)\qquad \frac{\Gamma;-\vdash v:\sigma\quad \Gamma,x:\sigma;\Pi\vdash a:\tau}{\Gamma;\Pi\vdash a\langle x:=v\rangle:\tau}\ (Cut_2)$$

$$\frac{\Delta;-\vdash v:\sigma\quad \Gamma;\Pi\vdash a:\tau}{\Gamma;\Pi\vdash a\langle x:=v\rangle:\tau}\ (K\text{-}cut)$$
where $x\notin\Gamma$

Proof. See Lemma 2.4 (ii) of [2]. □

Table 4 presents the rules of the type assignment system $\overline{\lambda}\mathbf{x}_\cap$, which is based on two kinds of judgements: $\Gamma;-\vdash t:\tau$ and $\Gamma;\sigma\vdash l:\tau$. We use $\Gamma;\Pi\vdash a:\tau$ to denote both kinds of judgements, with Π being zero or one type. The type σ in $\Gamma;\sigma\vdash l:\tau$ represents the type of a head variable to be attached to the list l. In other words, σ is the type of the hole of l, since l can be viewed as a context with a hole in the position of the head variable (cf. the comparison between pure terms and pure λ-terms in the previous section). So in the rule $(L\to)$, the hole with type τ in the right premiss is replaced, in the conclusion, by the hole with type $\sigma\to\tau$ applied to the term t which is typed with σ in the left premiss. In the rule (Cut_1), we use the notation al, which is read as the $\overline{\lambda}\mathbf{x}$-term obtained by filling the hole of l with a term or another context a.

In the following we show some lemmas on properties of the system $\overline{\lambda}\mathbf{x}_\cap$.

Lemma 14.

1. *If $\Gamma;\Pi\vdash a:\tau$, $y\notin\Gamma$ and $y\notin FV(a)$ then $\Gamma[y/x];\Pi\vdash a[y/x]:\tau$.*
2. *If $\Gamma;\Pi\vdash a:\tau$ and $x\notin\Gamma$ then $\Gamma,x:\sigma;\Pi\vdash a:\tau$.*
3. *If $\Gamma,x:\sigma;\Pi\vdash a:\tau$ and $x\notin FV(a)$ then $\Gamma;\Pi\vdash a:\tau$.*

Proof. By induction on the structure of derivations. □

Lemma 15. *If $\Gamma;\sigma\vdash[\,]:\tau$ then $\sigma\leqslant\tau$.*

Proof. By induction on the derivation of $\Gamma;\sigma\vdash[\,]:\tau$. □

Lemma 16. *If $\Gamma, x : \sigma; \Pi \vdash a : \tau$ and $\rho \leqslant \sigma$ then $\Gamma, x : \rho; \Pi \vdash a : \tau$.*

Proof. By induction on the derivation of $\Gamma, x : \sigma; \Pi \vdash a : \tau$. □

Lemma 17. *If $\Gamma; - \vdash xl : \tau$ then there exists σ such that $x : \sigma \in \Gamma$.*

Proof. By induction on the derivation of $\Gamma; - \vdash xl : \tau$. □

Now we state a precise form of Generation Lemma for the system $\overline{\lambda}\mathtt{x}_\cap$.

Lemma 18 (Generation Lemma).

1. *$\Gamma, x : \sigma; - \vdash xl : \tau$ if and only if $\Gamma, x : \sigma; \sigma \vdash l : \tau$.*
2. *$\Gamma; - \vdash \lambda x.t : \tau$ if and only if there exist $\sigma_1, \ldots, \sigma_n, \rho_1, \ldots, \rho_n$ $(n \geqslant 1)$ such that $\cap_{\underline{n}}(\sigma_i \to \rho_i) \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma, x : \sigma_i; - \vdash t : \rho_i$.*
3. *$\Gamma; - \vdash tl : \tau$ if and only if there exists σ such that $\Gamma; - \vdash t : \sigma$ and $\Gamma; \sigma \vdash l : \tau$.*
4. *$\Gamma; - \vdash t\langle x := v\rangle : \tau$ if and only if either*
   (a) there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma, x : \sigma; - \vdash t : \tau$, or
   (b) $\Gamma; - \vdash t : \tau$ $(x \notin \Gamma)$ and there exist Δ, σ such that $\Delta; - \vdash v : \sigma$.
5. *$\Gamma; \rho \vdash t :: l : \tau$ if and only if there exist σ, ν such that $\rho \leqslant \sigma \to \nu$, $\Gamma; - \vdash t : \sigma$ and $\Gamma; \nu \vdash l : \tau$.*
6. *$\Gamma; \rho \vdash ll' : \tau$ if and only if there exists σ such that $\Gamma; \rho \vdash l : \sigma$ and $\Gamma; \sigma \vdash l' : \tau$.*
7. *$\Gamma; \rho \vdash l\langle x := v\rangle : \tau$ if and only if either*
   (a) there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma, x : \sigma; \rho \vdash l : \tau$, or
   (b) $\Gamma; \rho \vdash l : \tau$ $(x \notin \Gamma)$ and there exist Δ, σ such that $\Delta; - \vdash v : \sigma$.

Proof. See Appendix A. □

We are now in a position to show that the system $\overline{\lambda}\mathtt{x}_\cap$ satisfies the subject reduction property.

Theorem 6 (Subject Reduction). *If $a \to_{\overline{\lambda}\mathtt{x}} b$ and $\Gamma; \Pi \vdash a : \tau$ then $\Gamma; \Pi \vdash b : \tau$.*

Proof. See Appendix A. □

Finally we need the following proposition which states that the bijection Ψ preserves the typability of pure λ-terms and pure terms in $\mathcal{T}_{\overline{\lambda}\mathtt{x}}$.

Proposition 6. *For any pure λ-term M, M is typable in $\lambda\mathtt{x}_\cap$ (without using (Cut) and $(K\text{-}cut)$) if and only if $\Psi(M)$ is typable in $\overline{\lambda}\mathtt{x}_\cap$.*

Proof. See Appendix A. □

Now we can prove one direction of the characterization theorem of strongly normalizing $\overline{\lambda}\mathtt{x}$-terms.

Theorem 7. *If a is typable in the system $\overline{\lambda}\mathtt{x}_\cap$ then $a \in \mathcal{SN}_{\overline{\lambda}\mathtt{x}}$.*

Proof. By Theorem 5, it suffices to show that if a is typable in the system $\overline{\lambda}\mathtt{x}_\cap$ then $a \in \overline{\lambda}\mathtt{x}^{<\infty}$. Let a be typable in $\overline{\lambda}\mathtt{x}_\cap$ and b be any subterm of a. Then b is also typable in $\overline{\lambda}\mathtt{x}_\cap$, and by Theorem 6, so is $\mathtt{x}(b)$. If $\mathtt{x}(b) \in \mathcal{T}_{\overline{\lambda}\mathtt{x}}$ then by Proposition 6, $\Psi^{-1}(\mathtt{x}(b))$ is typable, and so $\Psi^{-1}(\mathtt{x}(b)) \in \mathcal{SN}_\beta$ by the result in [19]. Hence by the definition of β-reduction on pure terms, we have $\mathtt{x}(b) \in \mathcal{SN}_\beta$. If $\mathtt{x}(b) \in \mathcal{L}_{\overline{\lambda}\mathtt{x}}$ then each element $u \in \mathcal{T}_{\overline{\lambda}\mathtt{x}}$ of the list $\mathtt{x}(b)$ must be reduced independently, and so $\mathtt{x}(b) \in \mathcal{SN}_\beta$. Thus we obtain $a \in \overline{\lambda}\mathtt{x}^{<\infty}$. ☐

The above theorem extends the results of [11, 15] where strong normalization is proved for $\overline{\lambda}\mathtt{x}$-terms typed with simple types. Our system $\overline{\lambda}\mathtt{x}_\cap$ is, in fact, able to type all strongly normalizing $\overline{\lambda}\mathtt{x}$-terms. To show that, we introduce again the notion of typably decent terms.

Definition 8 (Typably decent terms). *A $\overline{\lambda}\mathtt{x}$-term a is said to be* typably decent *if for every substitution $\langle x := v\rangle$ occurring in a, v is typable in $\overline{\lambda}\mathtt{x}_\cap$.*

Lemma 19. *If a is typably decent, $a \to_\mathtt{x} b$ and $\Gamma;\Pi \vdash b : \tau$, then $\Gamma;\Pi \vdash a : \tau$.*

Proof. See Appendix A. ☐

Lemma 20. *If a is typably decent and $a \to_\mathtt{x} b$, then b is typably decent.*

Proof. Similar to the proof of Lemma 9. ☐

Lemma 21. *If a is typably decent, $a \xrightarrow{*}_\mathtt{x} b$ and $\Gamma;\Pi \vdash b : \tau$, then $\Gamma;\Pi \vdash a : \tau$.*

Proof. By induction on the length of the reduction steps of $a \xrightarrow{*}_\mathtt{x} b$, using Lemmas 19 and 20. ☐

Now we can prove the converse of Theorem 7.

Theorem 8. *If $a \in \mathcal{SN}_{\overline{\lambda}\mathtt{x}}$ then a is typable in the system $\overline{\lambda}\mathtt{x}_\cap$.*

Proof. By induction on the structure of a. Suppose that $a \in \mathcal{SN}_{\overline{\lambda}\mathtt{x}}$. Then for every substitution $\langle x := v\rangle$ occurring in a, $v \in \mathcal{SN}_{\overline{\lambda}\mathtt{x}}$, so by the induction hypothesis, v is typable in $\overline{\lambda}\mathtt{x}_\cap$. Hence a is typably decent. On the other hand, since $a \in \mathcal{SN}_{\overline{\lambda}\mathtt{x}}$, we have $\mathtt{x}(a) \in \mathcal{SN}_{\overline{\lambda}\mathtt{x}}$, so by Lemma 11, $\mathtt{x}(a) \in \mathcal{SN}_\beta$.

If $\mathtt{x}(a) \in \mathcal{T}_{\overline{\lambda}\mathtt{x}}$ then by the definition of β-reduction on pure terms, we have $\Psi^{-1}(\mathtt{x}(a)) \in \mathcal{SN}_\beta$, so by the result in [19], $\Psi^{-1}(\mathtt{x}(a))$ is typable in $\lambda\mathtt{x}_\cap$ (without using (*Cut*) and (*K-cut*)). Hence by Proposition 6, $\mathtt{x}(a)$ is typable in $\overline{\lambda}\mathtt{x}_\cap$. Therefore by Lemma 21, a is typable in $\overline{\lambda}\mathtt{x}_\cap$.

If $\mathtt{x}(a) \in \mathcal{L}_{\overline{\lambda}\mathtt{x}}$ then for any variable y, $y\mathtt{x}(a)$ is a pure term in $\mathcal{T}_{\overline{\lambda}\mathtt{x}}$. Since $\mathtt{x}(a) \in \mathcal{SN}_\beta$, we have $y\mathtt{x}(a) \in \mathcal{SN}_\beta$, so by the above argument, $y\mathtt{x}(a)$ is typable in $\overline{\lambda}\mathtt{x}_\cap$. Hence $\mathtt{x}(a)$ is typable in $\overline{\lambda}\mathtt{x}_\cap$, and by Lemma 21, a is typable in $\overline{\lambda}\mathtt{x}_\cap$. ☐

6 Conclusion

In this paper, we presented a method for lifting to explicit substitution calculi characterizations of the strongly normalizing terms of λ-calculus by means of intersection type systems. In the first half of the paper, we gave a simple proof of characterizing the strongly normalizing terms of $\lambda\mathtt{x}$-calculus by an intersection type system in [16]. In the latter half of the paper, we characterized the strongly normalizing terms of the explicit substitution calculus of [11] by a new intersection type system based on sequent calculus.

A challenging problem is to characterize the strongly normalizing terms of $\overline{\lambda}\mu\tilde{\mu}$-calculus [7] with explicit substitutions explored in [18]. For $\overline{\lambda}\mu\tilde{\mu}$-calculus without explicit substitutions, Dougherty et al. [10] studied characterization of the strongly normalizing terms, using intersection and union types. For having the subject reduction property, they imposed some restrictions on types of variables, whereas we introduced subtyping rules with a (semantically justified) pre-ordering. Another direction for further work is to extend our technique to explicit substitution calculi with composition and/or equations like the calculus in [13].

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A Omitted Proofs

In this appendix we supply some proofs that are omitted in the main body.

Lemma 7.

2. $\Gamma \vdash (\lambda y.M)\langle x := N\rangle : \tau$ *if and only if* $\Gamma \vdash \lambda y.M\langle x := N\rangle : \tau$ $(y \notin FV(N))$.

Proof.

($\Rightarrow$) Let $\Gamma \vdash (\lambda y.M)\langle x := N\rangle : \tau$. Then by Lemma 6 (5), we have two cases:

(i) there exists σ such that $\Gamma \vdash N : \sigma$ and $\Gamma, x : \sigma \vdash \lambda y.M : \tau$. In this case, by Lemma 6 (3), there exist $\nu_1, \ldots, \nu_n, \rho_1, \ldots, \rho_n$ such that $\cap_{\underline{n}}(\nu_i \to \rho_i) \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma, x : \sigma, y : \nu_i \vdash M : \rho_i$. On the other hand, by Lemma 2 (2), for each $i \in \underline{n}$, we have $\Gamma, y : \nu_i \vdash N : \sigma$. Then by the rule (Cut), for each $i \in \underline{n}$, $\Gamma, y : \nu_i \vdash M\langle x := N\rangle : \rho_i$, so by the rule $(\to I)$, $\Gamma \vdash \lambda y.M\langle x := N\rangle : \nu_i \to \rho_i$. Hence by the rule $(\cap I)$ and Lemma 3, we obtain $\Gamma \vdash \lambda y.M\langle x := N\rangle : \tau$.

(ii) $\Gamma \vdash \lambda y.M : \tau$ $(x \notin \Gamma)$ and there exist Δ, σ such that $\Delta \vdash N : \sigma$. This case is proved similarly to the case (i), using (K-*cut*) instead of (Cut).

($\Leftarrow$) Let $\Gamma \vdash \lambda y.M\langle x := N\rangle : \tau$ $(y \notin FV(N))$. Then by Lemma 6 (3), there exist $\nu_1, \ldots, \nu_n, \rho_1, \ldots, \rho_n$ such that $\cap_{\underline{n}}(\nu_i \to \rho_i) \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma, y : \nu_i \vdash M\langle x := N\rangle : \rho_i$. By Lemma 6 (5), for each $i \in \underline{n}$, there are two possibilities:

(i) there exists σ such that $\Gamma, y : \nu_i \vdash N : \sigma$ and $\Gamma, y : \nu_i, x : \sigma \vdash M : \rho_i$. In this case, by the rule $(\to I)$, we have $\Gamma, x : \sigma \vdash \lambda y.M : \nu_i \to \rho_i$. On the other hand, since $y \notin FV(N)$, we have $\Gamma \vdash N : \sigma$ by Lemma 2 (3). Hence by the rule (Cut), we get $\Gamma \vdash (\lambda y.M)\langle x := N\rangle : \nu_i \to \rho_i$.

(ii) $\Gamma, y : \nu_i \vdash M : \rho_i$ $(x \notin \Gamma, y : \nu_i)$ and there exist Δ, σ such that $\Delta \vdash N : \sigma$. Similarly to the case (i), we get $\Gamma \vdash (\lambda y.M)\langle x := N\rangle : \nu_i \to \rho_i$ using (K-*cut*) instead of (Cut).

Hence by the rule $(\cap I)$, we have $\Gamma \vdash (\lambda y.M)\langle x := N\rangle : \cap_{\underline{n}}(\nu_i \to \rho_i)$, and by Lemma 3, we obtain $\Gamma \vdash (\lambda y.M)\langle x := N\rangle : \tau$. ☐

Lemma 18 (Generation Lemma).

1. $\Gamma, x : \sigma; - \vdash xl : \tau$ *if and only if* $\Gamma, x : \sigma; \sigma \vdash l : \tau$.
2. $\Gamma; - \vdash \lambda x.t : \tau$ *if and only if there exist* $\sigma_1, \ldots, \sigma_n, \rho_1, \ldots, \rho_n$ $(n \geqslant 1)$ *such that* $\bigcap_{\underline{n}}(\sigma_i \to \rho_i) \leqslant \tau$ *and, for all* $i \in \underline{n}$, $\Gamma, x : \sigma_i; - \vdash t : \rho_i$.
3. $\Gamma; - \vdash tl : \tau$ *if and only if there exists* σ *such that* $\Gamma; - \vdash t : \sigma$ *and* $\Gamma; \sigma \vdash l : \tau$.
4. $\Gamma; - \vdash t\langle x := v\rangle : \tau$ *if and only if either*
   (a) there exists σ *such that* $\Gamma; - \vdash v : \sigma$ *and* $\Gamma, x : \sigma; - \vdash t : \tau$, *or*
   (b) $\Gamma; - \vdash t : \tau$ $(x \notin \Gamma)$ *and there exist* Δ, σ *such that* $\Delta; - \vdash v : \sigma$.
5. $\Gamma; \rho \vdash t :: l : \tau$ *if and only if there exist* σ, ν *such that* $\rho \leqslant \sigma \to \nu$, $\Gamma; - \vdash t : \sigma$ *and* $\Gamma; \nu \vdash l : \tau$.
6. $\Gamma; \rho \vdash ll' : \tau$ *if and only if there exists* σ *such that* $\Gamma; \rho \vdash l : \sigma$ *and* $\Gamma; \sigma \vdash l' : \tau$.
7. $\Gamma; \rho \vdash l\langle x := v\rangle : \tau$ *if and only if either*
   (a) there exists σ *such that* $\Gamma; - \vdash v : \sigma$ *and* $\Gamma, x : \sigma; \rho \vdash l : \tau$, *or*
   (b) $\Gamma; \rho \vdash l : \tau$ $(x \notin \Gamma)$ *and there exist* Δ, σ *such that* $\Delta; - \vdash v : \sigma$.

Proof. The right-to-left implications immediately follow from the typing rules. The converses are proved by induction on the structure of derivations. First we consider the case in part 5 where the last applied rule in the derivation is $(R\cap)$:

$$\frac{\Gamma; \rho \vdash t :: l : \tau_1 \quad \Gamma; \rho \vdash t :: l : \tau_2}{\Gamma; \rho \vdash t :: l : \tau_1 \cap \tau_2}\ (R\cap)$$

In this case, by the induction hypothesis, there are $\sigma_1, \nu_1, \sigma_2, \nu_2$ such that $\rho \leqslant \sigma_1 \to \nu_1$, $\Gamma; - \vdash t : \sigma_1$, $\Gamma; \nu_1 \vdash l : \tau_1$, $\rho \leqslant \sigma_2 \to \nu_2$, $\Gamma; - \vdash t : \sigma_2$ and $\Gamma; \nu_2 \vdash l : \tau_2$. Then by the rule $(R\cap)$, we have $\Gamma; - \vdash t : \sigma_1 \cap \sigma_2$, and by the rules $(L\leqslant)$ and $(R\cap)$, we have $\Gamma; \nu_1 \cap \nu_2 \vdash l : \tau_1 \cap \tau_2$. Since $\rho \leqslant (\sigma_1 \to \nu_1) \cap (\sigma_2 \to \nu_2) \leqslant (\sigma_1 \cap \sigma_2) \to (\nu_1 \cap \nu_2)$ by Lemma 12, we can take $\sigma = \sigma_1 \cap \sigma_2$ and $\nu = \nu_1 \cap \nu_2$.

Next we consider the case in part 7 where the last applied rule in the derivation is $(R\cap)$:

$$\frac{\Gamma; \rho \vdash l\langle x := v\rangle : \tau_1 \quad \Gamma; \rho \vdash l\langle x := v\rangle : \tau_2}{\Gamma; \rho \vdash l\langle x := v\rangle : \tau_1 \cap \tau_2}\ (R\cap)$$

In this case, by the induction hypothesis, we have the following four possibilities:

(i) there exist σ_1, σ_2 such that $\Gamma; - \vdash v : \sigma_1$, $\Gamma, x : \sigma_1; \rho \vdash l : \tau_1$, $\Gamma; - \vdash v : \sigma_2$ and $\Gamma, x : \sigma_2; \rho \vdash l : \tau_2$. In this case, by Lemma 16, $\Gamma, x : \sigma_1 \cap \sigma_2; \rho \vdash l : \tau_1$ and $\Gamma, x : \sigma_1 \cap \sigma_2; \rho \vdash l : \tau_2$, so by the rule $(R\cap)$, $\Gamma, x : \sigma_1 \cap \sigma_2; \rho \vdash l : \tau_1 \cap \tau_2$. On the other hand, by the rule $(R\cap)$, we have $\Gamma; - \vdash v : \sigma_1 \cap \sigma_2$. Hence (a) holds for $\sigma = \sigma_1 \cap \sigma_2$.

(ii) there exist $\sigma_1, \Delta, \sigma_2$ such that $\Gamma; - \vdash v : \sigma_1$, $\Gamma, x : \sigma_1; \rho \vdash l : \tau_1$, $\Gamma; \rho \vdash l : \tau_2$ and $\Delta; - \vdash v : \sigma_2$. In this case, by Lemma 14 (2), we have $\Gamma, x : \sigma_1; \rho \vdash l : \tau_2$, so by the rule $(R\cap)$, $\Gamma, x : \sigma_1; \rho \vdash l : \tau_1 \cap \tau_2$. Hence (a) holds for $\sigma = \sigma_1$.

(iii) there exist $\Delta, \sigma_1, \sigma_2$ such that $\Gamma; \rho \vdash l : \tau_1$, $\Delta; - \vdash v : \sigma_1$, $\Gamma; - \vdash v : \sigma_2$ and $\Gamma, x : \sigma_2; \rho \vdash l : \tau_2$. This case is proved similarly to the case (ii).

(iv) there exist $\Delta, \sigma_1, \Delta', \sigma_2$ such that $\Gamma; \rho \vdash l : \tau_1$ $(x \notin \Gamma)$, $\Delta; - \vdash v : \sigma_1$, $\Gamma; \rho \vdash l : \tau_2$ and $\Delta'; - \vdash v : \sigma_2$. In this case, by the rule $(R\cap)$, we have $\Gamma; \rho \vdash l : \tau_1 \cap \tau_2$. Hence (b) holds. □

Theorem 6 (Subject Reduction). *If* $a \to_{\overline{\lambda}\mathrm{x}} b$ *and* $\Gamma; \Pi \vdash a : \tau$ *then* $\Gamma; \Pi \vdash b : \tau$.

Proof. By induction on the reduction relation $\to_{\overline{\lambda}\mathrm{x}}$. First we consider the cases where the reduction is at the root. Most of these cases proceed analogously to the corresponding cut-elimination steps for typing derivations of simply typed terms. (For reference, we show the cut-elimination steps in Appendix B.)

($Beta$): Let $\Gamma; - \vdash (\lambda x.t)(u :: l) : \tau$. Then by Lemma 18 (3), there exist σ such that $\Gamma; - \vdash \lambda x.t : \sigma$ and $\Gamma; \sigma \vdash u :: l : \tau$. By Lemma 18 (2), there exist $\nu_1, \ldots, \nu_n, \rho_1, \ldots, \rho_n$ such that $\cap_{\underline{n}}(\nu_i \to \rho_i) \leqslant \sigma$ and, for all $i \in \underline{n}$, $\Gamma, x : \nu_i; - \vdash t : \rho_i$. On the other hand, by Lemma 18 (5), there exist μ, δ such that $\sigma \leqslant \mu \to \delta$, $\Gamma; - \vdash u : \mu$ and $\Gamma; \delta \vdash l : \tau$. From $\cap_{\underline{n}}(\nu_i \to \rho_i) \leqslant \sigma$ and $\sigma \leqslant \mu \to \delta$, we have $\cap_{\underline{n}}(\nu_i \to \rho_i) \leqslant \mu \to \delta$, and so by Lemma 13, there exist $i_1, \ldots, i_k \in \underline{n}$ such that $\mu \leqslant \nu_{i_1} \cap \cdots \cap \nu_{i_k}$ and $\rho_{i_1} \cap \cdots \cap \rho_{i_k} \leqslant \delta$. Since $\Gamma, x : \nu_i; - \vdash t : \rho_i$ for all $i \in \underline{n}$, we have $\Gamma, x : \nu_{i_1} \cap \cdots \cap \nu_{i_k}; - \vdash t : \rho_{i_1} \cap \cdots \cap \rho_{i_k}$ by the rules $(L \leqslant)$ and $(R \cap)$, and hence we get $\Gamma, x : \mu; - \vdash t : \delta$. Now by applying the rules (Cut_2) and (Cut_1) to $\Gamma; - \vdash u : \mu$, $\Gamma, x : \mu; - \vdash t : \delta$ and $\Gamma; \delta \vdash l : \tau$, we obtain $\Gamma; - \vdash t\langle x := u\rangle l : \tau$.

($1a$): Let $\Gamma; \rho \vdash []l : \tau$. Then by Lemma 18 (6), there exists σ such that $\Gamma; \rho \vdash [] : \sigma$ and $\Gamma; \sigma \vdash l : \tau$. By Lemma 15, we have $\rho \leqslant \sigma$, and hence we obtain $\Gamma; \rho \vdash l : \tau$.

($1b$): Let $\Gamma; \rho \vdash (u :: l)l' : \tau$. Then by Lemma 18 (6), there exists σ such that $\Gamma; \rho \vdash u :: l : \sigma$ and $\Gamma; \sigma \vdash l' : \tau$. By Lemma 18 (5), there exist μ, ν such that $\rho \leqslant \mu \to \nu$, $\Gamma; - \vdash u : \mu$ and $\Gamma; \nu \vdash l : \sigma$. Now by the rule (Cut_1), we have $\Gamma; \nu \vdash ll' : \tau$, so by the rule $(L \to)$, we get $\Gamma; \mu \to \nu \vdash u :: (ll') : \tau$. Hence we obtain $\Gamma; \rho \vdash u :: (ll') : \tau$.

($2a$): Let $\Gamma; \rho \vdash []\langle x := v\rangle : \tau$. Then by Lemma 18 (7), either (i) there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma, x : \sigma; \rho \vdash [] : \tau$, or (ii) $\Gamma; \rho \vdash [] : \tau$ $(x \notin \Gamma)$ and there exist Δ, σ such that $\Delta; - \vdash v : \sigma$. In the case (i), we have $\Gamma; \rho \vdash [] : \tau$ by Lemma 14 (3).

($2b$): Let $\Gamma; \rho \vdash (u :: l)\langle x := v\rangle : \tau$. Then by Lemma 18 (7), we have two cases:

- (i) there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma, x : \sigma; \rho \vdash u :: l : \tau$. In this case, by Lemma 18 (5), there exist μ, ν such that $\rho \leqslant \mu \to \nu$, $\Gamma, x : \sigma; - \vdash u : \mu$ and $\Gamma, x : \sigma; \nu \vdash l : \tau$. Then by the rule (Cut_2), $\Gamma; - \vdash u\langle x := v\rangle : \mu$ and $\Gamma; \nu \vdash l\langle x := v\rangle : \tau$, so by the rule $(L \to)$, $\Gamma; \mu \to \nu \vdash u\langle x := v\rangle :: l\langle x := v\rangle : \tau$. Hence we obtain $\Gamma; \rho \vdash u\langle x := v\rangle :: l\langle x := v\rangle : \tau$.
- (ii) $\Gamma; \rho \vdash u :: l : \tau$ $(x \notin \Gamma)$ and there exist Δ, σ such that $\Delta; - \vdash v : \sigma$. This case is proved similarly to the case (i), using (K-cut) instead of (Cut_2).

($3a$): Let $\Gamma; - \vdash (xl)l' : \tau$. Then by Lemma 18 (3), there exists σ such that $\Gamma; - \vdash xl : \sigma$ and $\Gamma; \sigma \vdash l' : \tau$. By Lemma 17, $\Gamma = \Gamma', x : \rho$ for some Γ', ρ, so by Lemma 18 (1), $\Gamma', x : \rho; \rho \vdash l : \sigma$. By the rule (Cut_1), we have $\Gamma', x : \rho; \rho \vdash ll' : \tau$, and hence we obtain $\Gamma', x : \rho; - \vdash x(ll') : \tau$.

($3b$): Let $\Gamma; - \vdash (\lambda y.t)[] : \tau$. Then by Lemma 18 (3), there exists σ such that $\Gamma; - \vdash \lambda y.t : \sigma$ and $\Gamma; \sigma \vdash [] : \tau$. By Lemma 15, we have $\sigma \leqslant \tau$, and hence we obtain $\Gamma; - \vdash \lambda y.t : \tau$.

(4a): Let $\Gamma; - \vdash (yl)\langle x := v\rangle : \tau$ and $y \not\equiv x$. Then by Lemma 18 (4), either (i) there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma, x : \sigma; - \vdash yl : \tau$, or (ii) $\Gamma; - \vdash yl : \tau$ ($x \notin \Gamma$) and there exist Δ, σ such that $\Delta \vdash v : \sigma$. In each case, by Lemma 17, we have $\Gamma = \Gamma', y : \rho$ for some Γ', ρ. Then by Lemma 18 (1), (i) $\Gamma', x : \sigma, y : \rho; \rho \vdash l : \tau$, (ii) $\Gamma', y : \rho; \rho \vdash l : \tau$. Hence, using ($Cut_2$) or ($K$-$cut$) and ($Der$), we obtain $\Gamma; - \vdash yl\langle x := v\rangle : \tau$.

(4b): Let $\Gamma; - \vdash (xl)\langle x := v\rangle : \tau$. Then by Lemma 18 (4), either (i) there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma, x : \sigma; - \vdash xl : \tau$, or (ii) $\Gamma; - \vdash xl : \tau$ ($x \notin \Gamma$) and there exist Δ, σ such that $\Delta \vdash v : \sigma$. However, by Lemma 17, (ii) is impossible, and in the case (i), by Lemma 18 (1), $\Gamma, x : \sigma; \sigma \vdash l : \tau$. Then applying ($Cut_2$) and ($Cut_1$), we obtain $\Gamma; - \vdash vl\langle x := v\rangle : \tau$.

(4c): Let $\Gamma; - \vdash (\lambda y.t)\langle x := v\rangle : \tau$. Then by Lemma 18 (4), we have two cases:

- (i) there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma, x : \sigma; - \vdash \lambda y.t : \tau$. In this case, by Lemma 18 (2), there exist $\nu_1, \ldots, \nu_n, \rho_1, \ldots, \rho_n$ such that $\cap_{\underline{n}}(\nu_i \to \rho_i) \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma, x : \sigma, y : \nu_i; - \vdash t : \rho_i$. On the other hand, by Lemma 14 (2), we have $\Gamma, y : \nu_i; - \vdash v : \sigma$ for each $i \in \underline{n}$. Then by the rule (Cut_2), $\Gamma, y : \nu_i; - \vdash t\langle x := v\rangle : \rho_i$, and so by the rule ($R \to$), $\Gamma; - \vdash \lambda y.t\langle x := v\rangle : \nu_i \to \rho_i$ for each $i \in \underline{n}$. Hence by the rules ($R\cap$) and ($R \leqslant$), we obtain $\Gamma; - \vdash \lambda y.t\langle x := v\rangle : \tau$.
- (ii) $\Gamma; - \vdash \lambda y.t : \tau$ ($x \notin \Gamma$) and there exist Δ, σ such that $\Delta; - \vdash v : \sigma$. This case is proved similarly to the case (i), using (K-cut) instead of (Cut_2).

(5a): Easy, using Lemma 18 (6).

(5b): Let $\Gamma; \rho \vdash (ll')\langle x := v\rangle : \tau$. Then by Lemma 18 (7), we have two cases:

- (i) there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma, x : \sigma; \rho \vdash ll' : \tau$. In this case, by Lemma 18 (6), there exists μ such that $\Gamma, x : \sigma; \rho \vdash l : \mu$ and $\Gamma, x : \sigma; \mu \vdash l' : \tau$. Then by the rule ($Cut_2$), $\Gamma; \rho \vdash l\langle x := v\rangle : \mu$ and $\Gamma; \mu \vdash l'\langle x := v\rangle : \tau$, so by the rule ($Cut_1$), we obtain $\Gamma; \rho \vdash l\langle x := v\rangle l'\langle x := v\rangle : \tau$.
- (ii) $\Gamma; \rho \vdash ll' : \tau$ ($x \notin \Gamma$) and there exist Δ, σ such that $\Delta; - \vdash v : \sigma$. This case is proved similarly to the case (i), using (K-cut) instead of (Cut_2).

(5c): Easy, using Lemma 18 (6) and (3).

(5d): Let $\Gamma; - \vdash (tl)\langle x := v\rangle : \tau$. Then by Lemma 18 (4), we have two cases:

- (i) there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma, x : \sigma; - \vdash tl : \tau$. In this case, by Lemma 18 (3), there exists ρ such that $\Gamma, x : \sigma; - \vdash t : \rho$ and $\Gamma, x : \sigma; \rho \vdash l : \tau$. Then by the rule ($Cut_2$), $\Gamma; - \vdash t\langle x := v\rangle : \rho$ and $\Gamma; \rho \vdash l\langle x := v\rangle : \tau$, so by the rule ($Cut_1$), we obtain $\Gamma; - \vdash t\langle x := v\rangle l\langle x := v\rangle : \tau$.
- (ii) $\Gamma; - \vdash tl : \tau$ ($x \notin \Gamma$) and there exist Δ, σ such that $\Delta; - \vdash v : \sigma$. This case is proved similarly to the case (i), using (K-cut) instead of (Cut_2).

The cases where the reduction is not at the root are immediate by Lemma 18 and the induction hypothesis. □

Table 5. Translations Ψ and Θ

$$\begin{aligned} \Psi(x) &=_{def} x[] \\ \Psi(MN) &=_{def} \{\Psi(M)\}(\Psi(N) :: []) \\ \Psi(\lambda x.M) &=_{def} \lambda x.\Psi(M) \end{aligned} \qquad \begin{aligned} \Theta(xl) &=_{def} \Theta'(x, l) \\ \Theta(\lambda x.t) &=_{def} \lambda x.\Theta(t) \\ \Theta((\lambda x.t)(u :: l)) &=_{def} \Theta'(\lambda x.\Theta(t), u :: l) \\ \Theta'(M, []) &=_{def} M \\ \Theta'(M, u :: l) &=_{def} \Theta'(M\Theta(u), l) \end{aligned}$$

Proposition 6. *For any pure* λ*-term* M*,* M *is typable in* $\lambda\mathtt{x}_\cap$ *(without using* (*Cut*) *and* (*K-cut*)*) if and only if* $\Psi(M)$ *is typable in* $\overline{\lambda}\mathtt{x}_\cap$*.*

Proof. First we add the following rule (*Sub*) to the system $\lambda\mathtt{x}_\cap$ without (*Cut*) and (*K-cut*):

$$\frac{\Gamma \vdash M : \sigma \quad \sigma \leqslant \tau}{\Gamma \vdash M : \tau}\ (Sub)$$

where $\leqslant$ is the pre-ordering in Definition 7. It is known that typability of pure λ-terms does not change from that in $\lambda\mathtt{x}_\cap$ without (*Cut*) and (*K-cut*). We then show that, for any pure λ-term M, $\Gamma \vdash M : \tau$ if and only if $\Gamma; - \vdash \Psi(M) : \tau$.

To prove this, we need a formal definition of the bijection Ψ as shown in Table 5, which uses a meta-operation $\{\_\}\_$ defined in [11, 15]. Then the left-to-right implication is proved by induction on the derivation of $\Gamma \vdash M : \tau$. Here we consider the case where $M \equiv M_0 M_1$ and the last applied rule is $(\rightarrow E)$. Then we have

$$\dfrac{\dfrac{}{\Gamma; - \vdash \Psi(M_0) : \sigma \rightarrow \tau}\ \text{IH} \quad \dfrac{\dfrac{}{\Gamma; - \vdash \Psi(M_1) : \sigma}\ \text{IH} \quad \dfrac{}{\Gamma; \tau \vdash [] : \tau}\ (Ax)}{\Gamma; \sigma \rightarrow \tau \vdash \Psi(M_1) :: [] : \tau}\ (L \rightarrow)}{\Gamma; - \vdash \Psi(M_0)(\Psi(M_1) :: []) : \tau}\ (Cut_1)$$

As shown in [11, 15], $\Psi(M_0)(\Psi(M_1) :: []) \xrightarrow{*}_{\mathtt{x}} \{\Psi(M_0)\}(\Psi(M_1) :: []) \equiv \Psi(M_0 M_1)$, so we obtain $\Gamma; - \vdash \Psi(M_0 M_1) : \tau$ by Theorem 6.

For the right-to-left implication, it suffices to show the following by simultaneous induction on the derivations for pure terms (Θ is the inverse of Ψ):

(i) If $\Gamma; - \vdash t : \tau$ then $\Gamma \vdash \Theta(t) : \tau$.
(ii) If $\Gamma; \sigma \vdash l : \tau$ then $\Gamma \vdash \Theta'(M, l) : \tau$ for any M with $\Gamma \vdash M : \sigma$. □

Lemma 19. *If a is typably decent, $a \to_{\mathtt{x}} b$ and $\Gamma; \Pi \vdash b : \tau$, then $\Gamma; \Pi \vdash a : \tau$.*

Proof. By induction on the reduction relation $\to_{\mathtt{x}}$. First we consider the cases where the reduction is at the root.

- ($1a$): Let $\Gamma; \rho \vdash l : \tau$. Using the axiom $\Gamma; \rho \vdash [] : \rho$ and the rule (Cut_1), we obtain $\Gamma; \rho \vdash []l : \tau$.
- ($1b$): Let $\Gamma; \rho \vdash u :: (ll') : \tau$. Then by Lemma 18 (5), there exist σ, ν such that $\rho \leqslant \sigma \to \nu$, $\Gamma; - \vdash u : \sigma$ and $\Gamma; \nu \vdash ll' : \tau$. By Lemma 18 (6), there exist μ such that $\Gamma; \nu \vdash l : \mu$ and $\Gamma; \mu \vdash l' : \tau$. By the rule ($L \to$), we get $\Gamma; \sigma \to \nu \vdash u :: l : \mu$, and by the rules ($Cut_1$) and ($L \leqslant$), we obtain $\Gamma; \rho \vdash (u :: l)l' : \tau$.
- ($2a$): Let $\Gamma; \rho \vdash [] : \tau$. Since $a \equiv []\langle x := v\rangle$ for a fresh variable x and a is typably decent, there exist Δ, σ such that $\Delta; - \vdash v : \sigma$. Hence we obtain $\Gamma; \rho \vdash []\langle x := v\rangle : \tau$ by the rule (K-cut).
- ($2b$): Let $\Gamma; \rho \vdash u\langle x := v\rangle :: l\langle x := v\rangle : \tau$. Then by Lemma 18 (5), there exist σ, ν such that $\rho \leqslant \sigma \to \nu$, $\Gamma; - \vdash u\langle x := v\rangle : \sigma$ and $\Gamma; \nu \vdash l\langle x := v\rangle : \tau$. By Lemma 18 (4) and (7), there are four possibilities:
  - (i) there exist μ, δ such that $\Gamma; - \vdash v : \mu$, $\Gamma, x : \mu; - \vdash u : \sigma$, $\Gamma; - \vdash v : \delta$ and $\Gamma, x : \delta; \nu \vdash l : \tau$. In this case, by Lemma 16, $\Gamma, x : \mu \cap \delta; - \vdash u : \sigma$ and $\Gamma, x : \mu \cap \delta; \nu \vdash l : \tau$, so by the rule ($L \to$), $\Gamma, x : \mu \cap \delta; \sigma \to \nu \vdash u :: l : \tau$. On the other hand, by the rule ($R\cap$), we have $\Gamma; - \vdash v : \mu \cap \delta$. Hence by the rules ($Cut_2$) and ($L \leqslant$), we obtain $\Gamma; \rho \vdash (u :: l)\langle x := v\rangle : \tau$.
  - (ii) there exist μ, Δ, δ such that $\Gamma; - \vdash v : \mu$, $\Gamma, x : \mu; - \vdash u : \sigma$, $\Gamma; \nu \vdash l : \tau$ and $\Delta; - \vdash v : \delta$. In this case, by Lemma 14 (2), $\Gamma, x : \mu; \nu \vdash l : \tau$, so by the rule ($L \to$), $\Gamma, x : \mu; \sigma \to \nu \vdash u :: l : \tau$. Hence by the rules ($Cut_2$) and ($L \leqslant$), we obtain $\Gamma; \rho \vdash (u :: l)\langle x := v\rangle : \tau$.
  - (iii) there exist Δ, δ, μ such that $\Gamma; - \vdash u : \sigma$, $\Delta; - \vdash v : \delta$, $\Gamma; - \vdash v : \mu$ and $\Gamma, x : \mu; \nu \vdash l : \tau$. In this case, by Lemma 14 (2), $\Gamma, x : \mu; - \vdash u : \sigma$. The rest is proved similarly to the case (ii).
  - (iv) there exist $\Delta, \delta, \Delta', \mu$ such that $\Gamma; - \vdash u : \sigma$ ($x \notin \Gamma$), $\Delta; - \vdash v : \delta$, $\Gamma; \nu \vdash l : \tau$ and $\Delta'; - \vdash v : \mu$. In this case, by the rule ($L \to$), we have $\Gamma; \sigma \to \nu \vdash u :: l : \tau$. Hence by the rules ($K$-$cut$) and ($L \leqslant$), we obtain $\Gamma; \rho \vdash (u :: l)\langle x := v\rangle : \tau$.
- ($3a$): Let $\Gamma; - \vdash x(ll') : \tau$. In this case, by Lemma 17, we have $\Gamma = \Gamma', x : \sigma$ for some Γ', σ. Then by Lemma 18 (1), $\Gamma', x : \sigma; \sigma \vdash ll' : \tau$, and by Lemma 18 (6), there exists ρ such that $\Gamma', x : \sigma; \sigma \vdash l : \rho$ and $\Gamma', x : \sigma; \rho \vdash l' : \tau$. By the rule ($Der$), we get $\Gamma', x : \sigma; - \vdash xl : \rho$, and by the rule ($Cut_1$), we obtain $\Gamma', x : \sigma; - \vdash (xl)l' : \tau$.
- ($3b$): Let $\Gamma; - \vdash \lambda y.t : \tau$. Using the axiom $\Gamma; \tau \vdash [] : \tau$ and the rule (Cut_1), we obtain $\Gamma; - \vdash (\lambda y.t)[] : \tau$.
- ($4a$): Let $\Gamma; - \vdash yl\langle x := v\rangle : \tau$ and $y \not\equiv x$. In this case, by Lemma 17, we have $\Gamma = \Gamma', y : \sigma$ for some Γ', σ. Then by Lemma 18 (1), $\Gamma', y : \sigma; \sigma \vdash l\langle x := v\rangle : \tau$. Now by Lemma 18 (7), we have two cases:
  - (i) there exists ρ such that $\Gamma', y : \sigma; - \vdash v : \rho$ and $\Gamma', y : \sigma, x : \rho; \sigma \vdash l : \tau$. In this case, by the rule (Der), $\Gamma', y : \sigma, x : \rho; - \vdash yl : \tau$. Hence by the rule ($Cut_2$), we obtain $\Gamma', y : \sigma; - \vdash (yl)\langle x := v\rangle : \tau$.

(ii) $\Gamma', y : \sigma; \sigma \vdash l : \tau$ ($x \notin \Gamma', y : \sigma$) and there exist Δ, ρ such that $\Delta; - \vdash v : \rho$. Similarly to the case (i), we obtain $\Gamma', y : \sigma; - \vdash (yl)\langle x := v\rangle : \tau$ using (K-cut) instead of (Cut_2).

($4b$): Let $\Gamma; - \vdash vl\langle x := v\rangle : \tau$. Then by Lemma 18 (3), there exists σ such that $\Gamma; - \vdash v : \sigma$ and $\Gamma; \sigma \vdash l\langle x := v\rangle : \tau$. By Lemma 18 (7), we have two cases:

(i) there exists ρ such that $\Gamma; - \vdash v : \rho$ and $\Gamma, x : \rho; \sigma \vdash l : \tau$. In this case, by Lemma 16 and the rule ($L \leqslant$), we have $\Gamma, x : \rho \cap \sigma; \rho \cap \sigma \vdash l : \tau$, so by the rule ($Der$), $\Gamma, x : \rho \cap \sigma; - \vdash xl : \tau$. On the other hand, by the rule ($R\cap$), we have $\Gamma; - \vdash v : \rho \cap \sigma$. Hence by the rule ($Cut_2$), we obtain $\Gamma; - \vdash (xl)\langle x := v\rangle : \tau$.

(ii) $\Gamma; \sigma \vdash l : \tau$ ($x \notin \Gamma$) and there exist Δ, ρ such that $\Delta; - \vdash v : \rho$. In this case, by Lemma 14 (2), $\Gamma, x : \sigma; \sigma \vdash l : \tau$, so by the rule ($Der$), $\Gamma, x : \sigma; - \vdash xl : \tau$. Hence by the rule ($Cut_2$), we get $\Gamma; - \vdash (xl)\langle x := v\rangle : \tau$.

($4c$): Let $\Gamma; - \vdash \lambda y.t\langle x := v\rangle : \tau$ ($y \notin FV(v)$). Then by Lemma 18 (2), there exist $\nu_1, \ldots, \nu_n, \rho_1, \ldots, \rho_n$ such that $\cap_{\underline{n}}(\nu_i \to \rho_i) \leqslant \tau$ and, for all $i \in \underline{n}$, $\Gamma, y : \nu_i; - \vdash t\langle x := v\rangle : \rho_i$. By Lemma 18 (4), for each $i \in \underline{n}$, there are two possibilities:

(i) there exists σ such that $\Gamma, y : \nu_i; - \vdash v : \sigma$ and $\Gamma, y : \nu_i, x : \sigma; - \vdash t : \rho_i$. In this case, by the rule ($R \to$), we have $\Gamma, x : \sigma; - \vdash \lambda y.t : \nu_i \to \rho_i$. On the other hand, since $y \notin FV(v)$, we have $\Gamma; - \vdash v : \sigma$ by Lemma 14 (3). Hence by the rule (Cut_2), we get $\Gamma; - \vdash (\lambda y.t)\langle x := v\rangle : \nu_i \to \rho_i$.

(ii) $\Gamma, y : \nu_i; - \vdash t : \rho_i$ ($x \notin \Gamma, y : \nu_i$) and there exist Δ, σ such that $\Delta; - \vdash v : \sigma$. Similarly to the case (i), we get $\Gamma; - \vdash (\lambda y.t)\langle x := v\rangle : \nu_i \to \rho_i$ using (K-cut) instead of (Cut_2).

Hence by the rule ($R\cap$), we have $\Gamma; - \vdash (\lambda y.t)\langle x := v\rangle : \cap_{\underline{n}}(\nu_i \to \rho_i)$, and by the rule ($L \leqslant$), we obtain $\Gamma; - \vdash (\lambda y.t)\langle x := v\rangle : \tau$.

($5a$): Easy, using Lemma 18 (6).

($5b$): Let $\Gamma; \rho \vdash l\langle x := v\rangle l'\langle x := v\rangle : \tau$. Then by Lemma 18 (6), there exists σ such that $\Gamma; \rho \vdash l\langle x := v\rangle : \sigma$ and $\Gamma; \sigma \vdash l'\langle x := v\rangle : \tau$. By Lemma 18 (7), there are four possibilities:

(i) there exist μ, δ such that $\Gamma; - \vdash v : \mu$, $\Gamma, x : \mu; \rho \vdash l : \sigma$, $\Gamma; - \vdash v : \delta$ and $\Gamma, x : \delta; \sigma \vdash l' : \tau$. In this case, by Lemma 16, $\Gamma, x : \mu \cap \delta; \rho \vdash l : \sigma$ and $\Gamma, x : \mu \cap \delta; \sigma \vdash l' : \tau$, so by the rule ($Cut_1$), $\Gamma, x : \mu \cap \delta; \rho \vdash ll' : \tau$. On the other hand, by the rule ($R\cap$), we have $\Gamma; - \vdash v : \mu \cap \delta$. Hence by the rule ($Cut_2$), we obtain $\Gamma; \rho \vdash (ll')\langle x := v\rangle : \tau$.

(ii) there exist μ, Δ, δ such that $\Gamma; - \vdash v : \mu$, $\Gamma, x : \mu; \rho \vdash l : \sigma$, $\Gamma; \sigma \vdash l' : \tau$ and $\Delta; - \vdash v : \delta$. In this case, by Lemma 14 (2), $\Gamma, x : \mu; \sigma \vdash l' : \tau$, so by the rule ($Cut_1$), $\Gamma, x : \mu; \rho \vdash ll' : \tau$. Hence by the rule ($Cut_2$), we obtain $\Gamma; \rho \vdash (ll')\langle x := v\rangle : \tau$.

(iii) there exist Δ, δ, μ such that $\Gamma; \rho \vdash l : \sigma$, $\Delta; - \vdash v : \delta$, $\Gamma; - \vdash v : \mu$ and $\Gamma, x : \mu; \sigma \vdash l' : \tau$. In this case, by Lemma 14 (2), $\Gamma, x : \mu; \rho \vdash l : \sigma$. The rest is proved similarly to the case (ii).

(iv) there exist $\Delta, \delta, \Delta', \mu$ such that $\Gamma; \rho \vdash l : \sigma$ ($x \notin \Gamma$), $\Delta; - \vdash v : \delta$, $\Gamma; \sigma \vdash l' : \tau$ and $\Delta'; - \vdash v : \mu$. In this case, by the rule (Cut_1), we have $\Gamma; \rho \vdash ll' : \tau$. Hence by the rule ($K$-$cut$), we get $\Gamma; \rho \vdash (ll')\langle x := v\rangle : \tau$.

($5c$): Easy, using Lemma 18 (3) and (6).

(5d): Let $\Gamma; - \vdash t\langle x := v\rangle l\langle x := v\rangle : \tau$. Then by Lemma 18 (3), there exists σ such that $\Gamma; - \vdash t\langle x := v\rangle : \sigma$ and $\Gamma; \sigma \vdash l\langle x := v\rangle : \tau$. By Lemma 18 (4) and (7), there are four possibilities:

(i) there exist ρ, ν such that $\Gamma; - \vdash v : \rho$, $\Gamma, x : \rho; - \vdash t : \sigma$, $\Gamma; - \vdash v : \nu$ and $\Gamma, x : \nu; \sigma \vdash l : \tau$. In this case, by Lemma 16, $\Gamma, x : \rho \cap \nu; - \vdash t : \sigma$ and $\Gamma, x : \rho \cap \nu; \sigma \vdash l : \tau$, so by the rule (Cut_1), $\Gamma, x : \rho \cap \nu; - \vdash tl : \tau$. On the other hand, by the rule $(R\cap)$, we have $\Gamma; - \vdash v : \rho \cap \nu$. Hence by the rule (Cut_2), we obtain $\Gamma; - \vdash (tl)\langle x := v\rangle : \tau$.

(ii) there exist ρ, Δ, ν such that $\Gamma; - \vdash v : \rho$, $\Gamma, x : \rho; - \vdash t : \sigma$, $\Gamma; \sigma \vdash l : \tau$ and $\Delta; - \vdash v : \nu$. In this case, by Lemma 14 (2), $\Gamma, x : \rho; \sigma \vdash l : \tau$, so by the rule (Cut_1), $\Gamma, x : \rho; - \vdash tl : \tau$. Hence by the rule (Cut_2), we obtain $\Gamma; - \vdash (tl)\langle x := v\rangle : \tau$.

(iii) there exist Δ, ν, ρ such that $\Gamma; - \vdash t : \sigma$, $\Delta; - \vdash v : \nu$, $\Gamma; - \vdash v : \rho$ and $\Gamma, x : \rho; \sigma \vdash l : \tau$. In this case, by Lemma 14 (2), $\Gamma, x : \rho; - \vdash t : \sigma$. Then similarly to the case (ii), we obtain $\Gamma; - \vdash (tl)\langle x := v\rangle : \tau$.

(iv) there exist $\Delta, \nu, \Delta', \rho$ such that $\Gamma; - \vdash t : \sigma$ $(x \notin \Gamma)$, $\Delta; - \vdash v : \nu$, $\Gamma; \sigma \vdash l : \tau$ and $\Delta'; - \vdash v : \rho$. In this case, by the rule (Cut_1), we have $\Gamma; - \vdash tl : \tau$. Hence by the rule ($K$-$cut$), we get $\Gamma; - \vdash (tl)\langle x := v\rangle : \tau$.

The cases where the reduction is not at the root are immediate by Lemma 18 and the induction hypothesis. $\square$

B Cut-Elimination Steps for Typing Derivations

In this appendix we display the cut-elimination steps corresponding to the reduction rules of $\overline{\lambda}$x-calculus for simply typed terms.

Cut-Elimination Steps for Typing Derivations

$(Beta)$ $\quad (\lambda x.t)(u :: l) \to t\langle x := u\rangle l$

$$\dfrac{\dfrac{\Gamma, x:\sigma; - \vdash t:\tau}{\Gamma; - \vdash \lambda x.t : \sigma\to\tau}\,(R\to) \qquad \dfrac{\Gamma; -\vdash u:\sigma \quad \Gamma;\tau\vdash l:\rho}{\Gamma;\sigma\to\tau\vdash u::l:\rho}\,(L\to)}{\Gamma; -\vdash (\lambda x.t)(u::l):\rho}\,(Cut_1)$$

$$\to \qquad \dfrac{\dfrac{\Gamma; -\vdash u:\sigma \quad \Gamma, x:\sigma; -\vdash t:\tau}{\Gamma; -\vdash t\langle x:=u\rangle : \tau}\,(Cut_2) \qquad \Gamma;\tau\vdash l:\rho}{\Gamma; -\vdash t\langle x:=u\rangle l:\rho}\,(Cut_1)$$

(1a) $\quad []l \to l$

$$\dfrac{\dfrac{}{\Gamma;\sigma\vdash []:\sigma}\,(Ax) \qquad \Gamma;\sigma\vdash l:\rho}{\Gamma;\sigma\vdash []l:\rho}\,(Cut_1) \qquad \to \qquad \Gamma;\sigma\vdash l:\rho$$

(1b) $\quad (u :: l)l' \to u :: (ll')$

$$\dfrac{\dfrac{\Gamma; -\vdash u:\sigma \quad \Gamma;\tau\vdash l:\rho}{\Gamma;\sigma\to\tau\vdash u::l:\rho}\,(L\to) \qquad \Gamma;\rho\vdash l':\nu}{\Gamma;\sigma\to\tau\vdash (u::l)l':\nu}\,(Cut_1)$$

$$\to \qquad \dfrac{\Gamma; -\vdash u:\sigma \qquad \dfrac{\Gamma;\tau\vdash l:\rho \quad \Gamma;\rho\vdash l':\nu}{\Gamma;\tau\vdash ll':\nu}\,(Cut_1)}{\Gamma;\sigma\to\tau\vdash u::(ll'):\nu}\,(L\to)$$

(2a) $\quad []\langle x := v\rangle \to []$

$$\dfrac{\Gamma; -\vdash v:\sigma \qquad \dfrac{}{\Gamma, x:\sigma;\tau\vdash []:\tau}\,(Ax)}{\Gamma;\tau\vdash []\langle x:=v\rangle:\tau}\,(Cut_2) \qquad \to \qquad \dfrac{}{\Gamma;\tau\vdash []:\tau}\,(Ax)$$

(2b) $\quad (u :: l)\langle x := v\rangle \to u\langle x := v\rangle :: l\langle x := v\rangle$

$$\dfrac{\Gamma; -\vdash v:\sigma \qquad \dfrac{\Gamma, x:\sigma; -\vdash u:\tau \quad \Gamma, x:\sigma;\rho\vdash l:\nu}{\Gamma, x:\sigma;\tau\to\rho\vdash u::l:\nu}\,(L\to)}{\Gamma;\tau\to\rho\vdash (u::l)\langle x:=v\rangle:\nu}\,(Cut_2)$$

$$\to \qquad \dfrac{\dfrac{\Gamma; -\vdash v:\sigma \quad \Gamma, x:\sigma; -\vdash u:\tau}{\Gamma; -\vdash u\langle x:=v\rangle:\tau}\,(Cut_2) \qquad \dfrac{\Gamma; -\vdash v:\sigma \quad \Gamma, x:\sigma;\rho\vdash l:\nu}{\Gamma;\rho\vdash l\langle x:=v\rangle:\nu}\,(Cut_2)}{\Gamma;\tau\to\rho\vdash u\langle x:=v\rangle :: l\langle x:=v\rangle:\nu}\,(L\to)$$

(3a) $(xl)l' \to x(ll')$

$$\dfrac{\dfrac{\Gamma, x:\sigma;\sigma \vdash l:\tau}{\Gamma, x:\sigma;- \vdash xl:\tau}\,(Der) \quad \Gamma, x:\sigma;\tau \vdash l':\rho}{\Gamma, x:\sigma;- \vdash (xl)l':\rho}\,(Cut_1)$$

$$\to \quad \dfrac{\dfrac{\Gamma, x:\sigma;\sigma \vdash l:\tau \quad \Gamma, x:\sigma;\tau \vdash l':\rho}{\Gamma, x:\sigma;\sigma \vdash ll':\rho}\,(Cut_1)}{\Gamma, x:\sigma;- \vdash x(ll'):\rho}\,(Der)$$

(3b) $(\lambda y.t)[] \to \lambda y.t$

$$\dfrac{\Gamma;- \vdash \lambda y.t:\sigma \quad \dfrac{}{\Gamma;\sigma \vdash []:\sigma}\,(Ax)}{\Gamma;- \vdash (\lambda y.t)[]:\sigma}\,(Cut_1) \quad \to \quad \Gamma;- \vdash \lambda y.t:\sigma$$

(4a) $(yl)\langle x := v\rangle \to yl\langle x := v\rangle \qquad (y \not\equiv x)$

$$\dfrac{\Gamma, y:\sigma;- \vdash v:\tau \quad \dfrac{\Gamma, x:\tau, y:\sigma;\sigma \vdash l:\rho}{\Gamma, x:\tau, y:\sigma;- \vdash yl:\rho}\,(Der)}{\Gamma, y:\sigma;- \vdash (yl)\langle x := v\rangle:\rho}\,(Cut_2)$$

$$\to \quad \dfrac{\dfrac{\Gamma, y:\sigma;- \vdash v:\tau \quad \Gamma, x:\tau, y:\sigma;\sigma \vdash l:\rho}{\Gamma, y:\sigma;\sigma \vdash l\langle x := v\rangle:\rho}\,(Cut_2)}{\Gamma, y:\sigma;- \vdash yl\langle x := v\rangle:\rho}\,(Der)$$

(4b) $(xl)\langle x := v\rangle \to vl\langle x := v\rangle$

$$\dfrac{\Gamma;- \vdash v:\sigma \quad \dfrac{\Gamma, x:\sigma;\sigma \vdash l:\tau}{\Gamma, x:\sigma;- \vdash xl:\tau}\,(Der)}{\Gamma;- \vdash (xl)\langle x := v\rangle:\tau}\,(Cut_2)$$

$$\to \quad \dfrac{\Gamma;- \vdash v:\sigma \quad \dfrac{\Gamma;- \vdash v:\sigma \quad \Gamma, x:\sigma;\sigma \vdash l:\tau}{\Gamma;\sigma \vdash l\langle x := v\rangle:\tau}\,(Cut_2)}{\Gamma;- \vdash vl\langle x := v\rangle:\tau}\,(Cut_1)$$

(4c) $(\lambda y.t)\langle x := v\rangle \to \lambda y.t\langle x := v\rangle$

$$\dfrac{\Gamma;- \vdash v:\sigma \quad \dfrac{\Gamma, x:\sigma, y:\tau;- \vdash t:\rho}{\Gamma, x:\sigma;- \vdash \lambda y.t:\tau \to \rho}\,(R \to)}{\Gamma;- \vdash (\lambda y.t)\langle x := v\rangle:\tau \to \rho}\,(Cut_2)$$

$$\to \quad \dfrac{\dfrac{\dfrac{}{\Gamma, y:\tau;- \vdash v:\sigma}\,\textit{Lemma 14 (2)} \quad \Gamma, x:\sigma, y:\tau;- \vdash t:\rho}{\Gamma, y:\tau;- \vdash t\langle x := v\rangle:\rho}\,(Cut_2)}{\Gamma;- \vdash \lambda y.t\langle x := v\rangle:\tau \to \rho}\,(R \to)$$

(5a) $(ll')l'' \to l(l'l'')$

$$\frac{\dfrac{\Gamma;\sigma \vdash l:\tau \quad \Gamma;\tau \vdash l':\rho}{\Gamma;\sigma \vdash ll':\rho}\,(Cut_1) \qquad \Gamma;\rho \vdash l'':\nu}{\Gamma;\sigma \vdash (ll')l'':\nu}\,(Cut_1)$$

$$\to \quad \frac{\Gamma;\sigma \vdash l:\tau \qquad \dfrac{\Gamma;\tau \vdash l':\rho \quad \Gamma;\rho \vdash l'':\nu}{\Gamma;\tau \vdash l'l'':\nu}\,(Cut_1)}{\Gamma;\sigma \vdash l(l'l''):\nu}\,(Cut_1)$$

(5b) $(ll')\langle x:=v\rangle \to l\langle x:=v\rangle l'\langle x:=v\rangle$

$$\frac{\Gamma;-\vdash v:\sigma \qquad \dfrac{\Gamma,x:\sigma;\tau \vdash l:\rho \quad \Gamma,x:\sigma;\rho \vdash l':\nu}{\Gamma,x:\sigma;\tau \vdash ll':\nu}\,(Cut_1)}{\Gamma;\tau \vdash (ll')\langle x:=v\rangle:\nu}\,(Cut_2)$$

$$\to \quad \frac{\dfrac{\Gamma;-\vdash v:\sigma \quad \Gamma,x:\sigma;\tau \vdash l:\rho}{\Gamma;\tau \vdash l\langle x:=v\rangle:\rho}\,(Cut_2) \qquad \dfrac{\Gamma;-\vdash v:\sigma \quad \Gamma,x:\sigma;\rho \vdash l':\nu}{\Gamma;\rho \vdash l'\langle x:=v\rangle:\nu}\,(Cut_2)}{\Gamma;\tau \vdash l\langle x:=v\rangle l'\langle x:=v\rangle:\nu}\,(Cut_1)$$

(5c) $(tl)l' \to t(ll')$

$$\frac{\dfrac{\Gamma;-\vdash t:\sigma \quad \Gamma;\sigma \vdash l:\tau}{\Gamma;-\vdash tl:\tau}\,(Cut_1) \qquad \Gamma;\tau \vdash l':\rho}{\Gamma;-\vdash (tl)l':\rho}\,(Cut_1)$$

$$\to \quad \frac{\Gamma;-\vdash t:\sigma \qquad \dfrac{\Gamma;\sigma \vdash l:\tau \quad \Gamma;\tau \vdash l':\rho}{\Gamma;\sigma \vdash ll':\rho}\,(Cut_1)}{\Gamma;-\vdash t(ll'):\rho}\,(Cut_1)$$

(5d) $(tl)\langle x:=v\rangle \to t\langle x:=v\rangle l\langle x:=v\rangle$

$$\frac{\Gamma;-\vdash v:\sigma \qquad \dfrac{\Gamma,x:\sigma;-\vdash t:\tau \quad \Gamma,x:\sigma;\tau \vdash l:\rho}{\Gamma,x:\sigma;-\vdash tl:\rho}\,(Cut_1)}{\Gamma;-\vdash (tl)\langle x:=v\rangle:\rho}\,(Cut_2)$$

$$\to \quad \frac{\dfrac{\Gamma;-\vdash v:\sigma \quad \Gamma,x:\sigma;-\vdash t:\tau}{\Gamma;-\vdash t\langle x:=v\rangle:\tau}\,(Cut_2) \qquad \dfrac{\Gamma;-\vdash v:\sigma \quad \Gamma,x:\sigma;\tau \vdash l:\rho}{\Gamma;\tau \vdash l\langle x:=v\rangle:\rho}\,(Cut_2)}{\Gamma;-\vdash t\langle x:=v\rangle l\langle x:=v\rangle:\rho}\,(Cut_1)$$