On a Local-Step Cut-Elimination Procedure for the Intuitionistic Sequent Calculus

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Abstract. In this paper we investigate, for intuitionistic implicational logic, the relationship between normalization in natural deduction and cut-elimination in a standard sequent calculus. First we identify a subset of proofs in the sequent calculus that correspond to proofs in natural deduction. Then we define a reduction relation on those proofs that exactly corresponds to normalization in natural deduction. The reduction relation is simulated soundly and completely by a cut-elimination procedure which consists of local proof transformations. It follows that the sequent calculus with our cut-elimination procedure is a proper extension that is conservative over natural deduction with normalization.

1 Introduction

In his seminal paper [6], Gentzen introduced natural deduction systems and sequent calculi for intuitionistic and classical logics, and proved the logical equivalence of the systems for each logic as well as the cut-elimination theorems for sequent calculi. In [10], Prawitz systematically studied normalization processes in natural deduction, and gave a construction of cut-free proofs from normal proofs for the purpose of proving the cut-elimination theorems through normalization results in natural deduction (Appendix A.3 of [10]). The construction by Prawitz differs from Gentzen's in that it assigns a *cut-free* proof to each normal proof in natural deduction. Under the Curry-Howard correspondence [8], the computational meaning of the cut-free proofs can be considered as the simply typed λ-terms corresponding to the normal proofs in natural deduction.

To obtain a meaningful Curry-Howard correspondence for sequent calculus, we therefore need to define a mapping from non-normal proofs in natural deduction to some proofs in sequent calculus, and investigate the relationship between normalization and cut-elimination processes. In particular, it is crucial to identify the cut-elimination procedure that corresponds to β-reduction in the simply typed λ-calculus. This would also open the way for understanding the computational or constructive meaning of various logics, including those logics which allow natural cut-elimination procedures in their sequent calculi but do not have an appropriate natural deduction system and normalization in it (e.g. modal logics and substructural logics).

In this paper, we define a mapping from proofs in natural deduction to a subset of proofs in a standard sequent calculus for intuitionistic logic. We syntactically characterize the image of the mapping, and show that it is a bijection between proofs in natural deduction and the subset of proofs in the sequent calculus. We also define a reduction relation on the subset of proofs, and show that it coincides with the β-reduction relation under the bijection. Thus the identification of β-reduction in the sequent calculus is achieved. The problem then reduces to investigating the relationship between the image of β-reduction and cut-elimination in the sequent calculus. In this paper we introduce a fairly standard cut-elimination procedure, except that in some circumstances it allows a cut to pass over another cut. Then the image of β-reduction is shown to be simulated by the cut-elimination procedure. It is also shown that the cut-elimination procedure is sound in regard to β-reduction, i.e., it does not break β-reducibility in the isomorphic image of natural deduction.

Allowing cuts to pass over other cuts is one of the criteria stated in [12, 13]. (It was also noted in [7, 2] for Herbelin's sequent calculus). Using the strong normalization result of a cut-elimination procedure that satisfies the criteria, Urban [12] proved strong normalization of the simply typed λ-calculus. Note however that inferring strong normalization of β-reduction and identifying β-reduction in a sequent calculus are different, and the latter is more appropriate to obtain a relevant Curry-Howard correspondence for sequent calculus. In [12], Urban also defined another cut-elimination procedure which consists of local proof transformations. For the cut-elimination procedure to satisfy the criteria, he introduced labelled cut-rules whose instances are allowed to pass over usual cuts. The cut-elimination procedure we introduce in this paper also consists of local proof transformations, but uses only one cut-rule. Instead of introducing labelled cut-rules and allowing any labelled cuts to pass over usual cuts, we derive minimal requirements on permutation of cuts for the simulation of β-reduction from a thorough analysis of our proof of the simulation.

For Herbelin-style sequent calculi (i.e., sequent calculi with stoup), it is known that there is an isomorphism between natural deduction and a fragment of those sequent calculi [4, 3, 5]. In such a system, one can distinguish cuts according to information on the stoups of sequents, and know in which case a cut should be allowed to pass over another cut. This is analogous to the situation using the labelled cuts mentioned above. Also, the proof terms for Herbelin-style sequent calculi are constructed involving an additional syntactic category called applicative context. In contrast, we establish an isomorphism using only proof terms for the standard sequent calculus, as found in [9].

The organization of the paper is as follows. In Section 2 we introduce sequent calculus and our cut-elimination procedure. In Section 3 we establish an isomorphism between a fragment of the sequent calculus and natural deduction. In Section 4 we discuss simulation of β-reduction by our cut-elimination procedure. In Section 5 we conclude and give suggestions for further work.

2 Sequent Calculus

In this section we introduce a term notation for proofs in a standard sequent calculus for intuitionistic implicational logic. Our cut-elimination procedure is represented as reduction rules for those terms. For proof terms and normalization in natural deduction, we use the ordinary simply typed λ-calculus (Appendix B).

First, the set of raw terms for sequent proofs is defined by the grammar: $t ::= x \mid \lambda x.t \mid \langle xt/x\rangle t \mid [t/x]t$ where x ranges over a denumerable set of variables. $\langle \_\_/\_\rangle\_$ and $[\_/\_]\_$ are 4-ary and 3-ary function symbols, respectively, and may be regarded as two kinds of explicit substitutions. We use letters x, y, z, w for variables and t, s, r, u for terms. The notions of free and bound variables are defined as usual, with an additional clause that the variable x in $\langle ys/x\rangle t$ or $[s/x]t$ binds the free occurrences of x in t. The set of free variables of a term t is denoted by $FV(t)$. We often use the notation $\langle \underline{x}s/y\rangle t$ to denote $\langle xs/y\rangle t$ if $x \notin FV(s) \cup FV(t)$. For such terms and variables, we define the notion of *fresh head variable* as follows: $FHV(x) = x$ and $FHV(\langle \underline{x}s/y\rangle t) = x$. The symbol $\equiv$ denotes syntactic equality modulo α-conversion.

The term assignment for sequent proofs of intuitionistic implicational logic is given in Table 1. We define a context, ranged over by Γ, as a finite set of pairs $\{x_1 : A_1, \ldots, x_n : A_n\}$ where the variables are pairwise distinct. The context $\Gamma, x : A$ denotes the union $\Gamma \cup \{x : A\}$, and $x \notin \Gamma$ means that x does not appear in Γ. For precise representation of proofs by terms, we should specify formulas on binders, but we will omit them for brevity. If $x \notin FV(s) \cup FV(t)$ in the term $\langle xs/y\rangle t$, we assume $x \notin \Gamma$ in the rule $L \supset$, which means the formula $A \supset B$ is introduced without implicit contraction.

The reduction rules in Table 1 define a cut-elimination procedure for sequent proofs (cf. Appendix A). The notion of cut-reduction is defined by the contextual closures of these reduction rules. We use $\to_{\mathrm{cut}}$ for one-step reduction, $\xrightarrow{+}_{\mathrm{cut}}$ for its transitive closure, and $\xrightarrow{*}_{\mathrm{cut}}$ for its reflexive transitive closure. These kinds of notations are also used for the notions of other reductions in this paper.

The reduction rules (1) through (5) correspond to cut-elimination steps that permute a cut upwards through its right subproof. Similarly, the rules (6) and (7) correspond to steps permuting a cut upwards through its left subproof. The rule (*Beta*) corresponds to the key-case which breaks a cut on an implication into two cuts on its subformulas. The rules ($Perm_1$) and ($Perm_2$) are the new rules introduced in this paper. They permute two cuts with some restrictions. In ($Perm_1$), the left rule over the lower cut is another cut, and the right rules over both cuts must be $L \supset$ that introduces the cut-formula without implicit contraction. In ($Perm_2$), the right rule over the lower cut is another cut, which must construct a proof corresponding to a redex of the rule (*Beta*).

3 Pure Terms

Table 2 presents the syntax of *pure terms*, which are the subset of proof terms for sequent calculus that correspond to simply typed λ-terms, i.e., proof terms

Table 1. Sequent calculus

$$Ax\ \frac{}{\Gamma, x : A \vdash x : A} \qquad\qquad L\supset \frac{\Gamma \vdash s : A \quad \Gamma, y : B \vdash t : C}{\Gamma, x : A \supset B \vdash \langle xs/y\rangle t : C}\ y \notin \Gamma$$

$$R\supset \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \supset B}\ x \notin \Gamma \qquad Cut\ \frac{\Gamma \vdash s : A \quad \Gamma, x : A \vdash t : B}{\Gamma \vdash [s/x]t : B}\ x \notin \Gamma$$

$\langle \underline{x}s/y\rangle t$ is used for $\langle xs/y\rangle t$ when $x \notin FV(s) \cup FV(t)$. In that case we assume $x \notin \Gamma$ in the rule $L\supset$.

(1)	$[t/x]y \to y \quad (y \not\equiv x)$
(2)	$[t/x]x \to t$
(3)	$[s/x](\lambda y.t) \to \lambda y.[s/x]t$
(4)	$[r/z]\langle xs/y\rangle t \to \langle x([r/z]s)/y\rangle[r/z]t \quad (x \not\equiv z)$
(5)	$[r/x]\langle xs/y\rangle t \to [r/x]\langle \underline{x}([r/x]s)/y\rangle[r/x]t \quad$ if $x \in FV(s) \cup FV(t)$
(6)	$[z/x]\langle \underline{x}s/y\rangle t \to \langle zs/y\rangle t$
(7)	$[\langle xs/y\rangle t/z]r \to \langle xs/y\rangle[t/z]r$
$(Beta)$	$[\lambda z.r/x]\langle \underline{x}s/y\rangle t \to [[s/z]r/y]t$
$(Perm_1)$	$[[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to [r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t'$
$(Perm_2)$	$[u/w][\lambda z.r/x]\langle \underline{x}s/y\rangle t \to [[u/w](\lambda z.r)/x][u/w]\langle \underline{x}s/y\rangle t$

for natural deduction. We use letters $l, l', \ldots$ for variables or pure terms of the form $\langle \underline{y_1}s_1/y_2\rangle \ldots \langle \underline{y_{n-1}}s_{n-1}/y_n\rangle y_n$. The intuitive idea behind the syntax is that we translate a λ-term $xM_1M_2\ldots M_n$ to $\langle xs_1/y_1\rangle\langle \underline{y_1}s_2/y_2\rangle \ldots \langle \underline{y_{n-1}}s_n/y_n\rangle y_n$, and $(\lambda z.N)M_1M_2\ldots M_n$ to $[\lambda z.r/x]\langle \underline{x}s_1/y_1\rangle\langle \underline{y_1}s_2/y_2\rangle \ldots \langle \underline{y_{n-1}}s_n/y_n\rangle y_n$, where $N, M_1, M_2, \ldots, M_n$ are translated to $r, s_1, s_2, \ldots, s_n$ $(n \geqslant 1)$. So a non-normal proof corresponding to $(\lambda z.N)M_1M_2\ldots M_n$ is translated to a cut of the form:

$$\dfrac{\dfrac{\Gamma, z : A \vdash r : B}{\Gamma \vdash \lambda z.r : A \supset B}\ R\supset \quad \dfrac{\Gamma \vdash s_1 : A \quad \Gamma, y_1 : B \vdash l : C}{\Gamma, x : A \supset B \vdash \langle \underline{x}s_1/y_1\rangle l : C}\ L\supset}{\Gamma \vdash [\lambda z.r/x]\langle \underline{x}s_1/y_1\rangle l : C}\ Cut$$

where $l \equiv \langle \underline{y_1}s_2/y_2\rangle \ldots \langle \underline{y_{n-1}}s_n/y_n\rangle y_n$. We refer to this kind of cut as a β-cut.

For the definition of β-reduction on pure terms, we need a meta-substitution $\{\_/\_\}\_$. Since we cannot in general replace the free variable w in $\langle ws/y\rangle l$ by a pure term, we need a further meta-operation $\langle\{\_\}\_/\_\rangle\_$. The term $\langle\{u\}s/y\rangle l$ may be seen as an abbreviation for $\{u/w\}\langle \underline{w}s/y\rangle l$, and defined by induction on the structure of the pure term u. Note that if $FHV(l) = y$ then $FHV(\{u/w\}l) =$

Table 2. Pure terms

$t, s, r ::= x \mid \lambda x.t \mid \langle xs/y\rangle l \mid [\lambda z.r/x]\langle \underline{x}s/y\rangle l$
$l ::= x \mid \langle \underline{x}s/y\rangle l$
where $FHV(l) = y$ in the right hand sides.

$$(\beta) \qquad [\lambda z.r/x]\langle \underline{x}s/y\rangle l \;\to\; \{\{s/z\}r/y\}l$$

where

$$\begin{aligned}
\{u/w\}x &=_{def} x \qquad (x \not\equiv w)\\
\{u/w\}w &=_{def} u\\
\{u/w\}(\lambda x.t) &=_{def} \lambda x.\{u/w\}t\\
\{u/w\}\langle xs/y\rangle l &=_{def} \langle x(\{u/w\}s)/y\rangle\{u/w\}l \qquad (x \not\equiv w)\\
\{u/w\}\langle ws/y\rangle l &=_{def} \langle\pmb{\{}u\pmb{\}}(\{u/w\}s)/y\rangle\{u/w\}l\\
\{u/w\}[\lambda z.r/x]\langle \underline{x}s/y\rangle l &=_{def} [\lambda z.\{u/w\}r/x]\langle \underline{x}(\{u/w\}s)/y\rangle\{u/w\}l
\end{aligned}$$

$$\begin{aligned}
\langle\pmb{\{}x\pmb{\}}s'/y\rangle l' &=_{def} \langle xs'/y\rangle l'\\
\langle\pmb{\{}\lambda x.t\pmb{\}}s'/y\rangle l' &=_{def} [\lambda x.t/w]\langle \underline{w}s'/y\rangle l'\\
\langle\pmb{\{}\langle xs/w\rangle l\pmb{\}}s'/y\rangle l' &=_{def} \langle xs/w\rangle\langle\pmb{\{}l\pmb{\}}s'/y\rangle l'\\
\langle\pmb{\{}[\lambda z.r/x]\langle \underline{x}s/w\rangle l\pmb{\}}s'/y\rangle l' &=_{def} [\lambda z.r/x]\langle \underline{x}s/w\rangle\langle\pmb{\{}l\pmb{\}}s'/y\rangle l'
\end{aligned}$$

y $(w \not\equiv y)$ and $FHV(\langle\pmb{\{}l\pmb{\}}s'/w\rangle l') = y$, so the meta-operations are well-defined on pure terms. The operation $\langle\pmb{\{}\_\pmb{\}}\_/\_\rangle\_$ corresponds to the cut-elimination process where the right rule over the cut is $L\supset$ introducing the cut-formula without implicit contraction, and the cut is permuted upwards through its left subproof. (Meta-operations based on similar ideas are found in [12, 13, 4, 3], but not for the pure terms we defined above.)

In the rest of this section we establish an isomorphism between pure terms and λ-terms. For this we define the translations ρ and φ as shown in Table 3. (These translations turn out to preserve the types of terms; see Appendix B.) For normal proofs, the translation ρ agrees with Prawitz's translation [10]. While Prawitz defined his translation by induction on the structure of normal proofs, we define the translation ρ by induction on the usual syntax of λ-terms, since we need to show that the same translation also preserves β-reduction which is based on the meta-substitution defined along with the usual syntax of λ-terms.

Now we consider bijection and preservation of β-reduction in order.

3.1 ρ and φ Are Bijective

Lemma 1. *Let t, u be pure terms. If $x \notin FV(t)$, then $\{u/x\}t \equiv t$.*

Table 3. Translations ρ and φ

$\rho(x) =_{def} x$	$\varphi(x) =_{def} x$
$\rho(MN) =_{def} \langle\{\!\{\rho(M)\}\!\}\rho(N)/x\rangle x$	$\varphi(\lambda x.t) =_{def} \lambda x.\varphi(t)$
$\rho(\lambda x.M) =_{def} \lambda x.\rho(M)$	$\varphi(\langle xs/y\rangle l) =_{def} \{x\varphi(s)/y\}\varphi(l)$
	$\varphi([\lambda z.r/x]\langle \underline{x}s/y\rangle l) =_{def} \{(\lambda z.\varphi(r))\varphi(s)/y\}\varphi(l)$

Proof. By induction on the structure of t. □

Lemma 2. *Let u, s, l, s', l' be pure terms with $FHV(l) = w$ and $FHV(l') = y$. Then* $\langle\{\!\{\langle\{\!\{u\}\!\}s/w\rangle l\}\!\}s'/y\rangle l' \equiv \langle\{\!\{u\}\!\}s/w\rangle\langle\{\!\{l\}\!\}s'/y\rangle l'$.

Proof. By induction on the structure of u.

(a) $u \equiv x$. Then

$$\begin{aligned}\langle\{\!\{\langle\{\!\{x\}\!\}s/w\rangle l\}\!\}s'/y\rangle l' &\equiv \langle\{\!\{\langle xs/w\rangle l\}\!\}s'/y\rangle l'\\ &\equiv \langle xs/w\rangle\langle\{\!\{l\}\!\}s'/y\rangle l'\\ &\equiv \langle\{\!\{x\}\!\}s/w\rangle\langle\{\!\{l\}\!\}s'/y\rangle l'\end{aligned}$$

(b) $u \equiv \lambda z.r$. Then

$$\begin{aligned}\langle\{\!\{\langle\{\!\{\lambda z.r\}\!\}s/w\rangle l\}\!\}s'/y\rangle l' &\equiv \langle\{\!\{[\lambda z.r/x]\langle \underline{x}s/w\rangle l\}\!\}s'/y\rangle l'\\ &\equiv [\lambda z.r/x]\langle \underline{x}s/w\rangle\langle\{\!\{l\}\!\}s'/y\rangle l'\\ &\equiv \langle\{\!\{\lambda z.r\}\!\}s/w\rangle\langle\{\!\{l\}\!\}s'/y\rangle l'\end{aligned}$$

(c) $u \equiv \langle xr/z\rangle l''$. Then

$$\begin{aligned}\langle\{\!\{\langle\{\!\{\langle xr/z\rangle l''\}\!\}s/w\rangle l\}\!\}s'/y\rangle l' &\equiv \langle\{\!\{\langle xr/z\rangle\langle\{\!\{l''\}\!\}s/w\rangle l\}\!\}s'/y\rangle l'\\ &\equiv \langle xr/z\rangle\langle\{\!\{\langle\{\!\{l''\}\!\}s/w\rangle l\}\!\}s'/y\rangle l'\\ &\equiv \langle xr/z\rangle\langle\{\!\{l''\}\!\}s/w\rangle\langle\{\!\{l\}\!\}s'/y\rangle l' \qquad \text{(by IH)}\\ &\equiv \langle\{\!\{\langle xr/z\rangle l''\}\!\}s/w\rangle\langle\{\!\{l\}\!\}s'/y\rangle l'\end{aligned}$$

(d) $u \equiv [\lambda z.r/x]\langle \underline{x}t/y'\rangle l''$. Then

$$\begin{aligned}&\langle\{\!\{\langle\{\!\{[\lambda z.r/x]\langle \underline{x}t/y'\rangle l''\}\!\}s/w\rangle l\}\!\}s'/y\rangle l'\\ \equiv\ &\langle\{\!\{[\lambda z.r/x]\langle \underline{x}t/y'\rangle\langle\{\!\{l''\}\!\}s/w\rangle l\}\!\}s'/y\rangle l'\\ \equiv\ &[\lambda z.r/x]\langle \underline{x}t/y'\rangle\langle\{\!\{\langle\{\!\{l''\}\!\}s/w\rangle l\}\!\}s'/y\rangle l'\\ \equiv\ &[\lambda z.r/x]\langle \underline{x}t/y'\rangle\langle\{\!\{l''\}\!\}s/w\rangle\langle\{\!\{l\}\!\}s'/y\rangle l' \qquad \text{(by IH)}\\ \equiv\ &\langle\{\!\{[\lambda z.r/x]\langle \underline{x}t/y'\rangle l''\}\!\}s/w\rangle\langle\{\!\{l\}\!\}s'/y\rangle l'\end{aligned}$$

□

Lemma 3. *Let u, t, s', l' be pure terms with $FHV(l') = y$ and $x \not\equiv y$. Then* $\{u/x\}\langle\{\!\!\{t\}\!\!\}s'/y\rangle l' \equiv \langle\{\!\!\{\{u/x\}t\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l'$.

Proof. By induction on the structure of t.

(a) $t \equiv w$ ($w \not\equiv x$). Then

$$\begin{aligned}\{u/x\}\langle\{\!\!\{w\}\!\!\}s'/y\rangle l' &\equiv \{u/x\}\langle ws'/y\rangle l' \\ &\equiv \langle w\{u/x\}s'/y\rangle\{u/x\}l' \\ &\equiv \langle\{\!\!\{w\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l' \\ &\equiv \langle\{\!\!\{\{u/x\}w\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l'\end{aligned}$$

(b) $t \equiv x$. Then

$$\begin{aligned}\{u/x\}\langle\{\!\!\{x\}\!\!\}s'/y\rangle l' &\equiv \{u/x\}\langle xs'/y\rangle l' \\ &\equiv \langle\{\!\!\{u\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l' \\ &\equiv \langle\{\!\!\{\{u/x\}x\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l'\end{aligned}$$

(c) $t \equiv \lambda z.r$. Then

$$\begin{aligned}\{u/x\}\langle\{\!\!\{\lambda z.r\}\!\!\}s'/y\rangle l' &\equiv \{u/x\}[\lambda z.r/w]\langle \underline{w}s'/y\rangle l' \\ &\equiv [\lambda z.\{u/x\}r/w]\langle\underline{w}\{u/x\}s'/y\rangle\{u/x\}l' \\ &\equiv \langle\{\!\!\{\lambda z.\{u/x\}r\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l' \\ &\equiv \langle\{\!\!\{\{u/x\}(\lambda z.r)\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l'\end{aligned}$$

(d) $t \equiv \langle x's/w\rangle l$ ($x' \not\equiv x$). Then

$$\begin{aligned}&\{u/x\}\langle\{\!\!\{\langle x's/w\rangle l\}\!\!\}s'/y\rangle l' \\ &\equiv \{u/x\}\langle x's/w\rangle\langle\{\!\!\{l\}\!\!\}s'/y\rangle l' \\ &\equiv \langle x'\{u/x\}s/w\rangle\{u/x\}\langle\{\!\!\{l\}\!\!\}s'/y\rangle l' \\ &\equiv \langle x'\{u/x\}s/w\rangle\langle\{\!\!\{\{u/x\}l\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l' && \text{(by IH)} \\ &\equiv \langle\{\!\!\{\langle x'\{u/x\}s/w\rangle\{u/x\}l\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l' \\ &\equiv \langle\{\!\!\{\{u/x\}\langle x's/w\rangle l\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l'\end{aligned}$$

(e) $t \equiv \langle xs/w\rangle l$. Then

$$\begin{aligned}&\{u/x\}\langle\{\!\!\{\langle xs/w\rangle l\}\!\!\}s'/y\rangle l' \\ &\equiv \{u/x\}\langle xs/w\rangle\langle\{\!\!\{l\}\!\!\}s'/y\rangle l' \\ &\equiv \langle\{\!\!\{u\}\!\!\}\{u/x\}s/w\rangle\{u/x\}\langle\{\!\!\{l\}\!\!\}s'/y\rangle l' \\ &\equiv \langle\{\!\!\{u\}\!\!\}\{u/x\}s/w\rangle\langle\{\!\!\{\{u/x\}l\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l' && \text{(by IH)} \\ &\equiv \langle\{\!\!\{\langle\{\!\!\{u\}\!\!\}\{u/x\}s/w\rangle\{u/x\}l\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l' && \text{(by Lemma 2)} \\ &\equiv \langle\{\!\!\{\{u/x\}\langle xs/w\rangle l\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l'\end{aligned}$$

(f) $t \equiv [\lambda z.r/y']\langle \underline{y'}s/w\rangle l$. Then

$$\begin{aligned}
&\{u/x\}\langle\{\!\!\{[\lambda z.r/y']\langle \underline{y'}s/w\rangle l\}\!\!\}s'/y\rangle l' \\
&\equiv\ \{u/x\}[\lambda z.r/y']\langle \underline{y'}s/w\rangle\langle\{\!\!\{l\}\!\!\}s'/y\rangle l' \\
&\equiv\ [\lambda z.\{u/x\}r/y']\langle \underline{y'}\{u/x\}s/w\rangle\{u/x\}\langle\{\!\!\{l\}\!\!\}s'/y\rangle l' \\
&\equiv\ [\lambda z.\{u/x\}r/y']\langle \underline{y'}\{u/x\}s/w\rangle\langle\{\!\!\{\{u/x\}l\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l' \qquad \text{(by IH)} \\
&\equiv\ \langle\{\!\!\{[\lambda z.\{u/x\}r/y']\langle \underline{y'}\{u/x\}s/w\rangle\{u/x\}l\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l' \\
&\equiv\ \langle\{\!\!\{\{u/x\}[\lambda z.r/y']\langle \underline{y'}s/w\rangle l\}\!\!\}\{u/x\}s'/y\rangle\{u/x\}l'
\end{aligned}$$

□

Lemma 4. *Let N, M be λ-terms. Then $\rho(\{N/x\}M) \equiv \{\rho(N)/x\}\rho(M)$.*

Proof. By induction on the structure of M.

(a) $M \equiv y$ ($y \not\equiv x$). Then $\rho(\{N/x\}y) \equiv \rho(y) \equiv y \equiv \{\rho(N)/x\}y \equiv \{\rho(N)/x\}\rho(y)$.
(b) $M \equiv x$. Then $\rho(\{N/x\}x) \equiv \rho(N) \equiv \{\rho(N)/x\}x \equiv \{\rho(N)/x\}\rho(x)$.
(c) $M \equiv M_0M_1$. Then

$$\begin{aligned}
\rho(\{N/x\}(M_0M_1)) &\equiv\ \rho(\{N/x\}M_0\{N/x\}M_1) \\
&\equiv\ \langle\{\!\!\{\rho(\{N/x\}M_0)\}\!\!\}\rho(\{N/x\}M_1)/y\rangle y \\
&\equiv\ \langle\{\!\!\{\{\rho(N)/x\}\rho(M_0)\}\!\!\}\{\rho(N)/x\}\rho(M_1)/y\rangle y \qquad \text{(by IH)} \\
&\equiv\ \langle\{\!\!\{\{\rho(N)/x\}\rho(M_0)\}\!\!\}\{\rho(N)/x\}\rho(M_1)/y\rangle\{\rho(N)/x\}y \\
&\equiv\ \{\rho(N)/x\}\langle\{\!\!\{\rho(M_0)\}\!\!\}\rho(M_1)/y\rangle y \qquad \text{(by Lemma 3)} \\
&\equiv\ \{\rho(N)/x\}\rho(M_0M_1)
\end{aligned}$$

(d) $M \equiv \lambda y.M_0$. Then

$$\begin{aligned}
\rho(\{N/x\}(\lambda y.M_0)) &\equiv\ \rho(\lambda y.\{N/x\}M_0) \\
&\equiv\ \lambda y.\rho(\{N/x\}M_0) \\
&\equiv\ \lambda y.\{\rho(N)/x\}\rho(M_0) \qquad \text{(by IH)} \\
&\equiv\ \{\rho(N)/x\}(\lambda y.\rho(M_0)) \\
&\equiv\ \{\rho(N)/x\}\rho(\lambda y.M_0)
\end{aligned}$$

□

Lemma 5. *Let u, s', l' be pure terms with $FHV(l') = y$. Then $\varphi(\langle\{\!\!\{u\}\!\!\}s'/y\rangle l') \equiv \{\varphi(u)\varphi(s')/y\}\varphi(l')$.*

Proof. By induction on the structure of u.

(a) $u \equiv x$. Then

$$\begin{aligned}
\varphi(\langle\{\!\!\{x\}\!\!\}s'/y\rangle l') &\equiv\ \varphi(\langle xs'/y\rangle l') \\
&\equiv\ \{x\varphi(s')/y\}\varphi(l') \\
&\equiv\ \{\varphi(x)\varphi(s')/y\}\varphi(l')
\end{aligned}$$

(b) $u \equiv \lambda x.t$. Then

$$\begin{aligned}\varphi(\langle \{\!\{\lambda x.t\}\!\} s'/y\rangle l') &\equiv \varphi([\lambda x.t/w]\langle \underline{w} s'/y\rangle l') \\ &\equiv \{(\lambda x.\varphi(t))\varphi(s')/y\}\varphi(l') \\ &\equiv \{\varphi(\lambda x.t)\varphi(s')/y\}\varphi(l')\end{aligned}$$

(c) $u \equiv \langle xs/w\rangle l$. Then

$$\begin{aligned}\varphi(\langle \{\!\{\langle xs/w\rangle l\}\!\} s'/y\rangle l') &\equiv \varphi(\langle xs/w\rangle \langle \{\!\{l\}\!\} s'/y\rangle l') \\ &\equiv \{x\varphi(s)/w\}\varphi(\langle \{\!\{l\}\!\} s'/y\rangle l') \\ &\equiv \{x\varphi(s)/w\}\{\varphi(l)\varphi(s')/y\}\varphi(l') && \text{(by IH)} \\ &\equiv \{\{x\varphi(s)/w\}\varphi(l)\varphi(s')/y\}\varphi(l') && (*) \\ &\equiv \{\varphi(\langle xs/w\rangle l)\varphi(s')/y\}\varphi(l')\end{aligned}$$

where the step $(*)$ is established since we can assume $w \notin FV(s') \cup FV(l')$ and so $w \notin FV(\varphi(s')) \cup FV(\varphi(l'))$.

(d) $u \equiv [\lambda z.r/x]\langle \underline{x} s/w\rangle l$. Then

$$\begin{aligned}&\varphi(\langle \{\!\{[\lambda z.r/x]\langle \underline{x} s/w\rangle l\}\!\} s'/y\rangle l') \\ \equiv\ &\varphi([\lambda z.r/x]\langle \underline{x} s/w\rangle \langle \{\!\{l\}\!\} s'/y\rangle l') \\ \equiv\ &\{\lambda z.\varphi(r)\varphi(s)/w\}\varphi(\langle \{\!\{l\}\!\} s'/y\rangle l') \\ \equiv\ &\{\lambda z.\varphi(r)\varphi(s)/w\}\{\varphi(l)\varphi(s')/y\}\varphi(l') && \text{(by IH)} \\ \equiv\ &\{\{\lambda z.\varphi(r)\varphi(s)/w\}\varphi(l)\varphi(s')/y\}\varphi(l') && (*) \\ \equiv\ &\{\varphi([\lambda z.r/x]\langle \underline{x} s/w\rangle l)\varphi(s')/y\}\varphi(l')\end{aligned}$$

where the step $(*)$ is established since we can assume $w \notin FV(s') \cup FV(l')$ and so $w \notin FV(\varphi(s')) \cup FV(\varphi(l'))$. □

Lemma 6. *Let s, r, l be pure terms with $FHV(l) = y$. Then*

1. $\{\langle xs/w\rangle w/y\} l \equiv \langle xs/y\rangle l$,
2. $\{[\lambda z.r/x]\langle \underline{x} s/w\rangle w/y\} l \equiv [\lambda z.r/x]\langle \underline{x} s/y\rangle l$.

Proof. By cases on l. □

Proposition 1. *$\varphi \circ \rho = id$ and $\rho \circ \varphi = id$.*

Proof. The first part is by induction on the structure of λ-terms.

(a) $M \equiv x$. Then $(\varphi \circ \rho)(x) \equiv \varphi(x) \equiv x$.

(b) $M \equiv M_0 M_1$. Then

$$\begin{aligned}(\varphi \circ \rho)(M_0 M_1) &\equiv \varphi(\langle \{\!\{\rho(M_0)\}\!\} \rho(M_1)/x\rangle x) \\ &\equiv \{\varphi(\rho(M_0))\varphi(\rho(M_1))/x\}\varphi(x) && \text{(by Lemma 5)} \\ &\equiv \{M_0 M_1/x\} x && \text{(by IH)} \\ &\equiv M_0 M_1\end{aligned}$$

(c) $M \equiv \lambda x.M_0$. Then $(\varphi \circ \rho)(\lambda x.M_0) \equiv \varphi(\lambda x.\rho(M_0)) \equiv \lambda x.\varphi(\rho(M_0)) \overset{\text{IH}}{\equiv} \lambda x.M_0$.

The second part is by induction on the structure of pure terms.

(a) $t \equiv x$. Then $(\rho \circ \varphi)(x) \equiv \rho(x) \equiv x$.
(b) $t \equiv \lambda x.t_0$. Then $(\rho \circ \varphi)(\lambda x.t_0) \equiv \rho(\lambda x.\varphi(t_0)) \equiv \lambda x.\rho(\varphi(t_0)) \overset{\text{IH}}{\equiv} \lambda x.t_0$.
(c) $t \equiv \langle xs/y \rangle l$. Then

$$\begin{aligned}
(\rho \circ \varphi)(\langle xs/y \rangle l) &\equiv \rho(\{x\varphi(s)/y\}\varphi(l)) \\
&\equiv \{\rho(x\varphi(s))/y\}\rho(\varphi(l)) && \text{(by Lemma 4)} \\
&\equiv \{\rho(x\varphi(s))/y\}l && \text{(by IH)} \\
&\equiv \{\langle \{\!\!\{\rho(x)\}\!\!\}\rho(\varphi(s))/w \rangle w/y\}l \\
&\equiv \{\langle \{\!\!\{\rho(x)\}\!\!\}s/w \rangle w/y\}l && \text{(by IH)} \\
&\equiv \{\langle \{\!\!\{x\}\!\!\}s)/w \rangle w/y\}l \\
&\equiv \{\langle xs/w \rangle w/y\}l \\
&\equiv \langle xs/y \rangle l && \text{(by Lemma 6 (1))}
\end{aligned}$$

(d) $t \equiv [\lambda z.r/x]\langle \underline{x}s/y \rangle l$. Then

$$\begin{aligned}
(\rho \circ \varphi)([\lambda z.r/x]\langle \underline{x}s/y \rangle l) &\equiv \rho(\{(\lambda z.\varphi(r))\varphi(s)/y\}\varphi(l)) \\
&\equiv \{\rho((\lambda z.\varphi(r))\varphi(s))/y\}\rho(\varphi(l)) && \text{(by Lemma 4)} \\
&\equiv \{\rho((\lambda z.\varphi(r))\varphi(s))/y\}l && \text{(by IH)} \\
&\equiv \{\langle \{\!\!\{\rho(\lambda z.\varphi(r))\}\!\!\}\rho(\varphi(s))/w \rangle w/y\}l \\
&\equiv \{\langle \{\!\!\{\rho(\lambda z.\varphi(r))\}\!\!\}s/w \rangle w/y\}l && \text{(by IH)} \\
&\equiv \{\langle \{\!\!\{\lambda z.\rho(\varphi(r))\}\!\!\}s/w \rangle w/y\}l \\
&\equiv \{\langle \{\!\!\{\lambda z.r\}\!\!\}s/w \rangle w/y\}l && \text{(by IH)} \\
&\equiv \{[\lambda z.r/x]\langle \underline{x}s/w \rangle w/y\}l \\
&\equiv [\lambda z.r/x]\langle \underline{x}s/y \rangle l && \text{(by Lemma 6 (2))}
\end{aligned}$$

□

3.2 ρ and φ Preserve β-Reduction

Lemma 7. *Let u, t be pure terms. Then $\varphi(\{u/x\}t) \equiv \{\varphi(u)/x\}\varphi(t)$.*

Proof.

$$\begin{aligned}
\varphi(\{u/x\}t) &\equiv \varphi(\{\rho(\varphi(u))/x\}\rho(\varphi(t))) && \text{(by Proposition 1)} \\
&\equiv \varphi(\rho(\{\varphi(u)/x\}\varphi(t))) && \text{(by Lemma 4)} \\
&\equiv \{\varphi(u)/x\}\varphi(t) && \text{(by Proposition 1)}
\end{aligned}$$

□

Lemma 8. *Let u, u', s, s', l be pure terms with $FHV(l) = y$.*

1. *If $u \to_\beta u'$ then $\langle \{\!\!\{u\}\!\!\}s/y \rangle l \to_\beta \langle \{\!\!\{u'\}\!\!\}s/y \rangle l$.*
2. *If $s \to_\beta s'$ then $\langle \{\!\!\{u\}\!\!\}s/y \rangle l \to_\beta \langle \{\!\!\{u\}\!\!\}s'/y \rangle l$.*

Proof. 1. By induction on the structure of u.

(a) $u \equiv \lambda x.t$ and $t \to_\beta t'$. Then

$$\begin{aligned}\langle \{\!\{\lambda x.t\}\!\} s/y\rangle l &\equiv [\lambda x.t/w]\langle \underline{w} s/y\rangle l\\ &\to_\beta [\lambda x.t'/w]\langle \underline{w} s/y\rangle l\\ &\equiv \langle \{\!\{\lambda x.t'\}\!\} s/y\rangle l\end{aligned}$$

(b) $u \equiv \langle x s_0/w\rangle l_0$ and $s_0 \to_\beta s_0'$. Then

$$\begin{aligned}\langle \{\!\{\langle x s_0/w\rangle l_0\}\!\} s/y\rangle l &\equiv \langle x s_0/w\rangle \langle \{\!\{l_0\}\!\} s/y\rangle l\\ &\to_\beta \langle x s_0'/w\rangle \langle \{\!\{l_0\}\!\} s/y\rangle l\\ &\equiv \langle \{\!\{\langle x s_0'/w\rangle l_0\}\!\} s/y\rangle l\end{aligned}$$

(c) $u \equiv \langle x s_0/w\rangle l_0$ and $l_0 \to_\beta l_0'$. Then

$$\begin{aligned}\langle \{\!\{\langle x s_0/w\rangle l_0\}\!\} s/y\rangle l &\equiv \langle x s_0/w\rangle \langle \{\!\{l_0\}\!\} s/y\rangle l\\ &\to_\beta \langle x s_0/w\rangle \langle \{\!\{l_0'\}\!\} s/y\rangle l && \text{(by IH)}\\ &\equiv \langle \{\!\{\langle x s_0/w\rangle l_0'\}\!\} s/y\rangle l\end{aligned}$$

(d) $u \equiv [\lambda z.r/x]\langle \underline{x} s_0/w\rangle l_0$.

i. The β-reduction is at the root, i.e., $u' \equiv \{\{s_0/z\}r/w\}l_0$. Then

$$\begin{aligned}&\langle \{\!\{[\lambda z.r/x]\langle \underline{x} s_0/w\rangle l_0\}\!\} s/y\rangle l\\ &\equiv [\lambda z.r/x]\langle \underline{x} s_0/w\rangle \langle \{\!\{l_0\}\!\} s/y\rangle l\\ &\to_\beta \{\{s_0/z\}r/w\}\langle \{\!\{l_0\}\!\} s/y\rangle l\\ &\equiv \langle \{\!\{\{\{s_0/z\}r/w\}l_0\}\!\}\{\{s_0/z\}r/w\} s/y\rangle \{\{s_0/z\}r/w\} l\\ & && \text{(by Lemma 3)}\\ &\equiv \langle \{\!\{\{\{s_0/z\}r/w\}l_0\}\!\} s/y\rangle l && \text{(by Lemma 1)}\end{aligned}$$

ii. The β-reduction is internal, e.g., $l_0 \to_\beta l_0'$. Then

$$\begin{aligned}&\langle \{\!\{[\lambda z.r/x]\langle \underline{x} s_0/w\rangle l_0\}\!\} s/y\rangle l\\ &\equiv [\lambda z.r/x]\langle \underline{x} s_0/w\rangle \langle \{\!\{l_0\}\!\} s/y\rangle l\\ &\to_\beta [\lambda z.r/x]\langle \underline{x} s_0/w\rangle \langle \{\!\{l_0'\}\!\} s/y\rangle l && \text{(by IH)}\\ &\equiv \langle \{\!\{[\lambda z.r/x]\langle \underline{x} s_0/w\rangle l_0'\}\!\} s/y\rangle l\end{aligned}$$

The other cases are similar.

2. By induction on the structure of u, similarly to the item 1. □

Lemma 9. *Let l be a pure term with $FHV(l) = y$. Then y occurs exactly once in $\varphi(l)$.*

Proof. By induction on the structure of l. □

Theorem 1.

1. *For any λ-terms M, M', if $M \to_\beta M'$ then $\rho(M) \to_\beta \rho(M')$.*
2. *For any pure terms t, t', if $t \to_\beta t'$ then $\varphi(t) \to_\beta \varphi(t')$.*

Proof. 1. By induction on the structure of M.

(a) $M \equiv (\lambda x.M_0)M_1 \to_\beta \{M_1/x\}M_0 \equiv M'$. Then

$$\begin{aligned}
\rho((\lambda x.M_0)M_1) &\equiv \langle \{\!\{\rho(\lambda x.M_0)\}\!\}\rho(M_1)/y\rangle y \\
&\equiv \langle \{\!\{\lambda x.\rho(M_0)\}\!\}\rho(M_1)/y\rangle y \\
&\equiv [\lambda x.\rho(M_0)/w]\langle \underline{w}\rho(M_1)/y\rangle y \\
&\to_\beta \{\{\rho(M_1)/x\}\rho(M_0)/y\}y \\
&\equiv \{\rho(M_1)/x\}\rho(M_0) \\
&\equiv \rho(\{M_1/x\}M_0) && \text{(by Lemma 4)}
\end{aligned}$$

(b) $M \equiv M_0M_1$ and $M_0 \to_\beta M_0'$. By the induction hypothesis, $\rho(M_0) \to_\beta \rho(M_0')$. Hence

$$\begin{aligned}
\rho(M_0M_1) &\equiv \langle \{\!\{\rho(M_0)\}\!\}\rho(M_1)/y\rangle y \\
&\to_\beta \langle \{\!\{\rho(M_0')\}\!\}\rho(M_1)/y\rangle y && \text{(by Lemma 8 (1))} \\
&\equiv \rho(M_0'M_1)
\end{aligned}$$

(c) $M \equiv M_0M_1$ and $M_1 \to_\beta M_1'$. Similar, using Lemma 8 (2).

(d) $M \equiv \lambda x.M_0$ and $M_0 \to_\beta M_0'$. By the induction hypothesis, $\rho(M_0) \to_\beta \rho(M_0')$. Hence $\rho(\lambda x.M_0) \equiv \lambda x.\rho(M_0) \to_\beta \lambda x.\rho(M_0') \equiv \rho(\lambda x.M_0')$.

2. By induction on the structure of t.

(a) $t \equiv \lambda x.t_0$ and $t_0 \to_\beta t_0'$. By the induction hypothesis, $\varphi(t_0) \to_\beta \varphi(t_0')$. Hence $\varphi(\lambda x.t_0) \equiv \lambda x.\varphi(t_0) \to_\beta \lambda x.\varphi(t_0') \equiv \varphi(\lambda x.t_0')$.

(b) $t \equiv \langle xs/y\rangle l$ and $s \to_\beta s'$. By the induction hypothesis, $\varphi(s) \to_\beta \varphi(s')$. By Lemma 9, y has exactly one occurrence in $\varphi(l)$. Hence $\varphi(\langle xs/y\rangle l) \equiv \{x\varphi(s)/y\}\varphi(l) \to_\beta \{x\varphi(s')/y\}\varphi(l) \equiv \varphi(\langle xs'/y\rangle l)$.

(c) $t \equiv \langle xs/y\rangle l$ and $l \to_\beta l'$. Similar, using the induction hypothesis.

(d) $t \equiv [\lambda z.r/x]\langle \underline{x}s/y\rangle l$.

i. The β-reduction is at the root, i.e., $t' \equiv \{\{s/z\}r/y\}l$. Then

$$\begin{aligned}
\varphi([\lambda z.r/x]\langle \underline{x}s/y\rangle l) &\equiv \{(\lambda z.\varphi(r))\varphi(s)/y\}\varphi(l) \\
&\to_\beta \{\{\varphi(s)/z\}\varphi(r)/y\}\varphi(l) \\
&\equiv \{\varphi(\{s/z\}r)/y\}\varphi(l) && \text{(by Lemma 7)} \\
&\equiv \varphi(\{\{s/z\}r/y\}l) && \text{(by Lemma 7)}
\end{aligned}$$

ii. The β-reduction is internal, e.g., $r \to_\beta r'$. By the induction hypothesis, $\varphi(r) \to_\beta \varphi(r')$. Hence

$$\begin{aligned}
\varphi([\lambda z.r/x]\langle \underline{x}s/y\rangle l) &\equiv \{(\lambda z.\varphi(r))\varphi(s)/y\}\varphi(l) \\
&\to_\beta \{(\lambda z.\varphi(r'))\varphi(s)/y\}\varphi(l) \\
&\equiv \varphi([\lambda z.r'/x]\langle \underline{x}s/y\rangle l)
\end{aligned}$$

The other cases are similar, using the induction hypothesis. □

4 Simulation of β-Reduction

In this section we investigate the relation between $\to_{\text{cut}}$ and $\to_\beta$ on pure terms. This relates normalization in natural deduction and our cut-elimination procedure in the sequent calculus, since pure terms are the isomorphic image of proof terms for natural deduction, as shown in the previous section. It is important that on the one hand, the cut-reduction simulates β-reduction, and on the other hand, the cut-reduction is sound in regard to β-reduction (i.e., a pure term t reaches another pure term t' by the cut-reduction only if t is β-reducible to t').

The following lemmas show that the cut-reduction correctly simulates the meta-operations on pure terms.

Lemma 10. *Let u, s, l be pure terms with $FHV(l) = w$. Then $[u/y]\langle \underline{y}s/w\rangle l \overset{*}{\to}_{\text{cut}} \langle \{\!\!\{u\}\!\!\} s/w\rangle l$.*

Proof. By induction on the structure of u.

(a) $u \equiv x$. Then $[x/y]\langle \underline{y}s/w\rangle l \to_{\text{cut}} \langle xs/w\rangle l \equiv \langle \{\!\!\{x\}\!\!\} s/w\rangle l$.
(b) $u \equiv \lambda x.t$. Then $[\lambda x.t/y]\langle \underline{y}s/w\rangle l \equiv \langle \{\!\!\{\lambda x.t\}\!\!\} s/w\rangle l$.
(c) $u \equiv \langle xs'/w'\rangle l'$. Then

$$\begin{aligned}[\langle xs'/w'\rangle l'/y]\langle \underline{y}s/w\rangle l &\to_{\text{cut}} \langle xs'/w'\rangle [l'/y]\langle \underline{y}s/w\rangle l \\ &\overset{*}{\to}_{\text{cut}} \langle xs'/w'\rangle \langle \{\!\!\{l'\}\!\!\} s/w\rangle l && \text{(by IH)} \\ &\equiv \langle \{\!\!\{\langle xs'/w'\rangle l'\}\!\!\} s/w\rangle l\end{aligned}$$

(d) $u \equiv [\lambda z.r/x]\langle \underline{x}s'/w'\rangle l'$. Then

$$\begin{aligned}[[\lambda z.r/x]\langle \underline{x}s'/w'\rangle l'/y]\langle \underline{y}s/w\rangle l &\to_{\text{cut}} [\lambda z.r/x][\langle \underline{x}s'/w'\rangle l'/y]\langle \underline{y}s/w\rangle l \\ &\to_{\text{cut}} [\lambda z.r/x]\langle \underline{x}s'/w'\rangle [l'/y]\langle \underline{y}s/w\rangle l \\ &\overset{*}{\to}_{\text{cut}} [\lambda z.r/x]\langle \underline{x}s'/w'\rangle \langle \{\!\!\{l'\}\!\!\} s/w\rangle l && \text{(by IH)} \\ &\equiv \langle \{\!\!\{[\lambda z.r/x]\langle \underline{x}s'/w'\rangle l'\}\!\!\} s/w\rangle l\end{aligned}$$

□

Lemma 11. *Let u, t be pure terms. Then $[u/y]t \overset{*}{\to}_{\text{cut}} \{u/y\}t$.*

Proof. By induction on the structure of t.

(a) $t \equiv x$ ($x \not\equiv y$). Then $[u/y]x \to_{\text{cut}} x \equiv \{u/y\}x$.
(b) $t \equiv y$. Then $[u/y]y \to_{\text{cut}} u \equiv \{u/y\}y$.
(c) $t \equiv \lambda z.r$. Then $[u/y](\lambda z.r) \to_{\text{cut}} \lambda z.[u/y]r \overset{\text{IH}*}{\to}_{\text{cut}} \lambda z.\{u/y\}r \equiv \{u/y\}(\lambda z.r)$.
(d) $t \equiv \langle xs/w\rangle l$ ($x \not\equiv y$). Then

$$\begin{aligned}[u/y]\langle xs/w\rangle l &\to_{\text{cut}} \langle x([u/y]s)/w\rangle [u/y]l \\ &\overset{*}{\to}_{\text{cut}} \langle x(\{u/y\}s)/w\rangle \{u/y\}l && \text{(by IH)} \\ &\equiv \{u/y\}\langle xs/w\rangle l\end{aligned}$$

(e) $t \equiv \langle ys/w \rangle l$ ($y \in FV(s) \cup FV(l)$). Then

$$\begin{aligned}
[u/y]\langle ys/w\rangle l &\to_{\text{cut}} [u/y]\langle \underline{y}([u/y]s)/w\rangle [u/y]l \\
&\xrightarrow{*}_{\text{cut}} [u/y]\langle \underline{y}(\{u/y\}s)/w\rangle \{u/y\}l && \text{(by IH)} \\
&\xrightarrow{*}_{\text{cut}} \langle \{\!\!\{u\}\!\!\}(\{u/y\}s)/w\rangle \{u/y\}l && \text{(by Lemma 10)} \\
&\equiv \{u/y\}\langle ys/w\rangle l
\end{aligned}$$

(f) $t \equiv \langle \underline{y}s/w \rangle l$. Then

$$\begin{aligned}
[u/y]\langle \underline{y}s/w\rangle l &\xrightarrow{*}_{\text{cut}} \langle \{\!\!\{u\}\!\!\}s/w\rangle l && \text{(by Lemma 10)} \\
&\equiv \langle \{\!\!\{u\}\!\!\}(\{u/y\}s)/w\rangle \{u/y\}l && \text{(by Lemma 1)} \\
&\equiv \{u/y\}\langle \underline{y}s/w\rangle l
\end{aligned}$$

(g) $t \equiv [\lambda z.r/x]\langle \underline{x}s/w \rangle l$. Then

$$\begin{aligned}
[u/y][\lambda z.r/x]\langle \underline{x}s/w\rangle l &\to_{\text{cut}} [[u/y](\lambda z.r)/x][u/y]\langle \underline{x}s/w\rangle l \\
&\to_{\text{cut}} [\lambda z.[u/y]r/x][u/y]\langle \underline{x}s/w\rangle l \\
&\to_{\text{cut}} [\lambda z.[u/y]r/x]\langle \underline{x}([u/y]s)/w\rangle [u/y]l \\
&\xrightarrow{*}_{\text{cut}} [\lambda z.\{u/y\}r/x]\langle \underline{x}(\{u/y\}s)/w\rangle \{u/y\}l && \text{(by IH)} \\
&\equiv \{u/y\}[\lambda z.r/x]\langle \underline{x}s/w\rangle l
\end{aligned}$$

□

Now we are ready to show that the cut-reduction simulates β-reduction.

Theorem 2. *For any pure terms t, t', if $t \to_\beta t'$ then $t \xrightarrow{+}_{\text{cut}} t'$.*

Proof. By induction on the structure of t. We treat the case $t \equiv [\lambda z.r/x]\langle \underline{x}s/y \rangle l$, $t' \equiv \{\{s/z\}r/y\}l$. Then use $\to_{Beta}$ to create $[[s/z]r/y]l$, and use Lemma 11 to reach $\{\{s/z\}r/y\}l$. □

The proof of Theorem 2 indicates how to simulate normalization in natural deduction by our cut-elimination procedure in the sequent calculus. Specifically, a redex in natural deduction is translated into a β-cut corresponding to a *Beta*-redex $[\lambda z.r/x]\langle \underline{x}s/y \rangle l$. Then transformation is performed as in the case (*Beta*) of Appendix A to create the proof corresponding to $[[s/z]r/y]l$, followed by cut-elimination steps to reach the proof corresponding to $\{\{s/z\}r/y\}l$.

From the above proofs of Lemmas 10 and 11, we see that for simulation of β-reduction, the reduction rules (7) and ($Perm_1$) can be restricted to the following forms:

$$\begin{array}{ll}
(7') & [\langle xs/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to \langle xs/y\rangle [t/z]\langle \underline{z}s'/w\rangle t' \\
(Perm'_1) & [[\lambda z.r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to [\lambda z.r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t'
\end{array}$$

These reduction rules specify some strategies for cut-elimination. The rule $(7')$ makes the cut-elimination procedure first permute a cut upwards through its

Table 4. Translation $\widehat{\varphi}$

$$
\begin{aligned}
\widehat{\varphi}(x) &=_{def} x \\
\widehat{\varphi}(\lambda x.t) &=_{def} \lambda x.\widehat{\varphi}(t) \\
\widehat{\varphi}(\langle xs/y\rangle t) &=_{def} \{x\widehat{\varphi}(s)/y\}\widehat{\varphi}(t) \\
\widehat{\varphi}([s/x]t) &=_{def} \{\widehat{\varphi}(s)/x\}\widehat{\varphi}(t)
\end{aligned}
$$

right subproof and then through its left subproof. The rule ($Perm_1'$) restricts the cut-elimination procedure so that permutation of two cuts is allowed only when the upper cut corresponds to a *Beta*-redex.

Next we show that the cut-reduction is sound in regard to β-reduction. For this we define a translation $\widehat{\varphi}$ of all terms for sequent proofs into λ-terms, as shown in Table 4.

Proposition 2. *For any pure term t, $\widehat{\varphi}(t) \equiv \varphi(t)$.*

Proof. By induction on the structure of t. We treat the case $t \equiv [\lambda z.r/x]\langle \underline{x}s/y\rangle l$. Then

$$
\begin{aligned}
\widehat{\varphi}([\lambda z.r/x]\langle \underline{x}s/y\rangle l) &\equiv \{\widehat{\varphi}(\lambda z.r)/x\}\widehat{\varphi}(\langle \underline{x}s/y\rangle l) \\
&\equiv \{\lambda z.\widehat{\varphi}(r)/x\}\{x\widehat{\varphi}(s)/y\}\widehat{\varphi}(l) \\
&\equiv \{(\lambda z.\widehat{\varphi}(r))\widehat{\varphi}(s)/y\}\widehat{\varphi}(l) && (*) \\
&\equiv \{(\lambda z.\varphi(r))\varphi(s)/y\}\varphi(l) && \text{(by IH)} \\
&\equiv \varphi([\lambda z.r/x]\langle \underline{x}s/y\rangle l)
\end{aligned}
$$

where the step $(*)$ is established since $x \notin FV(s) \cup FV(l)$ and so $x \notin FV(\widehat{\varphi}(s)) \cup FV(\widehat{\varphi}(l))$. □

Now we show that the cut-reduction projects onto β-reduction.

Lemma 12 (Projection). *If $u \to_{\text{cut}} u'$, then $\widehat{\varphi}(u) \xrightarrow{*}_{\beta} \widehat{\varphi}(u')$.*

Proof. By induction on the structure of u. If the cut-reduction is not at the root then the lemma easily follows from the induction hypothesis. If the cut-reduction is at the root, for example, if $u \equiv [[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to_{\text{cut}} [r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \equiv u'$ then

$$
\begin{aligned}
\widehat{\varphi}([[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t') &\equiv \widehat{\varphi}([[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t') \\
&\equiv \{\widehat{\varphi}([r/x]\langle \underline{x}s/y\rangle t)/z\}\widehat{\varphi}(\langle \underline{z}s'/w\rangle t') \\
&\equiv \{\{\widehat{\varphi}(r)/x\}\widehat{\varphi}(\langle \underline{x}s/y\rangle t)/z\}\widehat{\varphi}(\langle \underline{z}s'/w\rangle t') \\
&\equiv \{\widehat{\varphi}(r)/x\}\{\widehat{\varphi}(\langle \underline{x}s/y\rangle t)/z\}\widehat{\varphi}(\langle \underline{z}s'/w\rangle t') && (*) \\
&\equiv \{\widehat{\varphi}(r)/x\}\widehat{\varphi}([\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t') \\
&\equiv \widehat{\varphi}([r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t')
\end{aligned}
$$

where the step $(*)$ is established since we can assume $x \notin FV(\langle \underline{z}s'/w\rangle t')$ and so $x \notin FV(\widehat{\varphi}(\langle \underline{z}s'/w\rangle t'))$. □

As a result, we have that $\to_{\text{cut}}$ is a sound refinement of $\to_\beta$.

Corollary 1. *For any pure terms t, t', if $t \xrightarrow{*}_{\text{cut}} t'$ then $t \xrightarrow{*}_\beta t'$.*

Proof. Suppose that $t \xrightarrow{*}_{\text{cut}} t'$. Then by Lemma 12, $\widehat{\varphi}(t) \xrightarrow{*}_\beta \widehat{\varphi}(t')$, so by Proposition 2, $\varphi(t) \xrightarrow{*}_\beta \varphi(t')$. Now by Theorem 1 (1), $\rho(\varphi(t)) \xrightarrow{*}_\beta \rho(\varphi(t'))$, and so by Proposition 1, we have $t \xrightarrow{*}_\beta t'$. □

5 Conclusion and Further Work

We have investigated the relationship between normalization in natural deduction and cut-elimination in a standard sequent calculus, using term notations for both systems. We have identified a subset of sequent proofs that correspond to simply typed λ-terms, and showed that the isomorphic image of β-reduction is simulated by our cut-elimination procedure. Since the cut-elimination procedure is also sound in regard to β-reduction, the sequent calculus can be considered as a conservative extension of natural deduction in both proofs and reduction relation. Moreover, we have derived minimal requirements for simulation of β-reduction by a local-step cut-elimination procedure, analyzing our proof of the simulation.

It is expected that our cut-elimination procedure satisfies the strong normalization property. However, unlike in the case using labelled cuts [12], a standard method for inferring strong normalization of explicit substitution calculus [1] is not directly applied to our case. One of the reasons is that the subcalculus (i.e., the reduction system without the rule (*Beta*)) is not confluent (e.g., the critical pair $w \leftarrow [\langle xs/y\rangle t/z]w \to \langle xs/y\rangle[t/z]w$ is not joinable). So we might need more powerful methods for proving strong normalization, which will be investigated in future work.

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A Cut-Elimination Steps for Sequent Proofs

In this appendix, we display the cut-elimination steps corresponding to the reduction rules in Table 1. First we need the following lemma.

Lemma 13. *Let $\Gamma \vdash t : A$ where x does not appear in Γ. Then $\Gamma, x : B \vdash t : A$.*

Proof. By induction on the structure of derivations. □

In the following $\boxed{x : B}$ means a declaration added by the above lemma.

Cut-Elimination Steps for Sequent Proofs

(1) $[t/x]y \to y \quad (y \not\equiv x)$

$$\dfrac{\Gamma, y : A \vdash t : B \quad \dfrac{}{\Gamma, x : B, y : A \vdash y : A}\,Ax}{\Gamma, y : A \vdash [t/x]y : A}\,Cut \quad \to \quad \dfrac{}{\Gamma, y : A \vdash y : A}\,Ax$$

(2) $[t/x]x \to t$

$$\dfrac{\Gamma \vdash t : A \quad \dfrac{}{\Gamma, x : A \vdash x : A}\,Ax}{\Gamma \vdash [t/x]x : A}\,Cut \quad \to \quad \Gamma \vdash t : A$$

(3) $[s/x](\lambda y.t) \to \lambda y.[s/x]t$

$$\dfrac{\Gamma \vdash s : A \quad \dfrac{\Gamma, x : A, y : B \vdash t : C}{\Gamma, x : A \vdash \lambda y.t : B \supset C}\ R\supset}{\Gamma \vdash [s/x](\lambda y.t) : B \supset C}\ Cut \quad \to \quad \dfrac{\dfrac{\Gamma, y : B \vdash s : A \quad \Gamma, x : A, y : B \vdash t : C}{\Gamma, y : B \vdash [s/x]t : C}\ Cut}{\Gamma \vdash \lambda y.[s/x]t : B \supset C}\ R\supset$$

(4) $[r/z]\langle xs/y\rangle t \to \langle x([r/z]s)/y\rangle [r/z]t \quad (x \not\equiv z)$

$$\dfrac{\Gamma, x : B \supset C \vdash r : A \quad \dfrac{\Gamma, z : A \vdash s : B \quad \Gamma, z : A, y : C \vdash t : D}{\Gamma, z : A, x : B \supset C \vdash \langle xs/y\rangle t : D}\ L\supset}{\Gamma, x : B \supset C \vdash [r/z]\langle xs/y\rangle t : D}\ Cut$$

$$\to \quad \dfrac{\dfrac{\Gamma, x : B \supset C \vdash r : A \quad \Gamma, z : A, x : B \supset C \vdash s : B}{\Gamma, x : B \supset C \vdash [r/z]s : B}\ Cut \quad \dfrac{\Gamma, x : B \supset C, y : C \vdash r : A \quad \Gamma, z : A, x : B \supset C, y : C \vdash t : D}{\Gamma, x : B \supset C, y : C \vdash [r/z]t : D}\ Cut}{\Gamma, x : B \supset C \vdash \langle x([r/z]s)/y\rangle [r/z]t : D}\ L\supset$$

(5) $[r/x]\langle xs/y\rangle t \to [r/x]\langle \underline{x}([r/x]s)/y\rangle [r/x]t \quad$ if $x \in FV(s) \cup FV(t)$

$$\dfrac{\Gamma \vdash r : A \supset B \quad \dfrac{\Gamma, x : A \supset B \vdash s : A \quad \Gamma, x : A \supset B, y : B \vdash t : C}{\Gamma, x : A \supset B \vdash \langle xs/y\rangle t : C}\ L\supset}{\Gamma \vdash [r/x]\langle xs/y\rangle t : C}\ Cut$$

$$\to \quad \dfrac{\Gamma \vdash r : A \supset B \quad \dfrac{\dfrac{\Gamma \vdash r : A \supset B \quad \Gamma, x : A \supset B \vdash s : A}{\Gamma \vdash [r/x]s : A}\ Cut \quad \dfrac{\Gamma, y : B \vdash r : A \supset B \quad \Gamma, x : A \supset B, y : B \vdash t : C}{\Gamma, y : B \vdash [r/x]t : C}\ Cut}{\Gamma, x : A \supset B \vdash \langle \underline{x}([r/x]s)/y\rangle [r/x]t : C}\ L\supset}{\Gamma \vdash [r/x]\langle \underline{x}([r/x]s)/y\rangle [r/x]t : C}\ Cut$$

(6) $[z/x]\langle \underline{x}s/y\rangle t \to \langle zs/y\rangle t$

$$\dfrac{\dfrac{}{\Gamma, z : A \supset B \vdash z : A \supset B}\ Ax \quad \dfrac{\Gamma, z : A \supset B \vdash s : A \quad \Gamma, z : A \supset B, y : B \vdash t : C}{\Gamma, z : A \supset B, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C}\ L\supset}{\Gamma, z : A \supset B \vdash [z/x]\langle \underline{x}s/y\rangle t : C}\ Cut$$

$$\to \quad \dfrac{\Gamma, z : A \supset B \vdash s : A \quad \Gamma, z : A \supset B, y : B \vdash t : C}{\Gamma, z : A \supset B \vdash \langle zs/y\rangle t : C}\ L\supset$$

(7) $[\langle xs/y\rangle t/z]r \to \langle xs/y\rangle [t/z]r$

$$\dfrac{\dfrac{\Gamma \vdash s : A \quad \Gamma, y : B \vdash t : C}{\Gamma, x : A \supset B \vdash \langle xs/y\rangle t : C}\ L\supset \quad \Gamma, x : A \supset B, z : C \vdash r : D}{\Gamma, x : A \supset B \vdash [\langle xs/y\rangle t/z]r : D}\ Cut$$

$$\to \quad \dfrac{\Gamma, x : A \supset B \vdash s : A \quad \dfrac{\Gamma, x : A \supset B, y : B \vdash t : C \quad \Gamma, x : A \supset B, y : B, z : C \vdash r : D}{\Gamma, x : A \supset B, y : B \vdash [t/z]r : D}\ Cut}{\Gamma, x : A \supset B \vdash \langle xs/y\rangle [t/z]r : D}\ L\supset$$

$(Beta)\quad [\lambda z.r/x]\langle \underline{x}s/y\rangle t \to [[s/z]r/y]t$

$$\dfrac{\dfrac{\Gamma, z : A \vdash r : B}{\Gamma \vdash \lambda z.r : A \supset B}\,R\supset \qquad \dfrac{\Gamma \vdash s : A \quad \Gamma, y : B \vdash t : C}{\Gamma, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C}\,L\supset}{\Gamma \vdash [\lambda z.r/x]\langle \underline{x}s/y\rangle t : C}\,Cut$$

$$\to \quad \dfrac{\dfrac{\Gamma \vdash s : A \quad \Gamma, z : A \vdash r : B}{\Gamma \vdash [s/z]r : B}\,Cut \qquad \Gamma, y : B \vdash t : C}{\Gamma \vdash [[s/z]r/y]t : C}\,Cut$$

$(Perm_1)\quad [[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to [r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t'$

$$\dfrac{\dfrac{\Gamma \vdash r : A \supset B \quad \Gamma, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C \supset D}{\Gamma \vdash [r/x]\langle \underline{x}s/y\rangle t : C \supset D}\,Cut \qquad \Gamma, z : C \supset D \vdash \langle \underline{z}s'/w\rangle t' : E}{\Gamma \vdash [[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' : E}\,Cut$$

$$\to \quad \dfrac{\Gamma \vdash r : A \supset B \qquad \dfrac{\Gamma, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C \supset D \quad \Gamma, x : A \supset B, z : C \supset D \vdash \langle \underline{z}s'/w\rangle t' : E}{\Gamma, x : A \supset B \vdash [\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' : E}\,Cut}{\Gamma \vdash [r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' : E}\,Cut$$

$(Perm_2)\quad [u/w][\lambda z.r/x]\langle \underline{x}s/y\rangle t \to [[u/w](\lambda z.r)/x][u/w]\langle \underline{x}s/y\rangle t$

$$\dfrac{\Gamma \vdash u : D \qquad \dfrac{\Gamma, w : D \vdash \lambda z.r : A \supset B \quad \Gamma, w : D, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C}{\Gamma, w : D \vdash [\lambda z.r/x]\langle \underline{x}s/y\rangle t : C}\,Cut}{\Gamma \vdash [u/w][\lambda z.r/x]\langle \underline{x}s/y\rangle t : C}\,Cut$$

$$\to \quad \dfrac{\dfrac{\Gamma \vdash u : D \quad \Gamma, w : D \vdash \lambda z.r : A \supset B}{\Gamma \vdash [u/w](\lambda z.r) : A \supset B}\,Cut \qquad \dfrac{\Gamma, x : A \supset B \vdash u : D \quad \Gamma, w : D, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C}{\Gamma, x : A \supset B \vdash [u/w]\langle \underline{x}s/y\rangle t : C}\,Cut}{\Gamma \vdash [[u/w](\lambda z.r)/x][u/w]\langle \underline{x}s/y\rangle t : C}\,Cut$$

B Natural Deduction

The raw terms for proofs in natural deduction are ordinary λ-terms, defined by the grammar: $M ::= x \mid MM \mid \lambda x.M$. The term assignment for proofs in natural deduction and the β-rule together with the definition of meta-substitution $\{\_/\_\}\_$ are given in Table 5.

In the following we verify that the translations in Tables 3 and 4 preserve the types of terms for proofs in natural deduction and in sequent calculus.

Proposition 3.

1. *For any λ-term M, if $\Gamma \vdash M : A$ then $\Gamma \vdash \rho(M) : A$.*
2. *For any pure term t, if $\Gamma \vdash t : A$ then $\Gamma \vdash \varphi(t) : A$.*
3. *For any proof term t for the sequent calculus, if $\Gamma \vdash t : A$ then $\Gamma \vdash \widehat{\varphi}(t) : A$.*

Table 5. Natural Deduction

$$Ax\ \frac{}{\Gamma, x : A \vdash x : A} \qquad \supset E\ \frac{\Gamma \vdash M : A \supset B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\supset I\ \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \supset B}\ x \notin \Gamma$$

(β) $\qquad (\lambda x.M)N \rightarrow \{N/x\}M$

where

$$\begin{aligned} \{N/x\}y &=_{def} y \qquad (y \not\equiv x) \\ \{N/x\}x &=_{def} N \\ \{N/x\}(MM') &=_{def} \{N/x\}M\{N/x\}M' \\ \{N/x\}(\lambda y.M) &=_{def} \lambda y.\{N/x\}M \end{aligned}$$

Proof. 1. By induction on the derivation of $\Gamma \vdash M : A$. Here we treat the case where $M \equiv M_0 M_1$ and the last rule applied is $\supset E$. Then we have

$$\dfrac{\dfrac{}{\Gamma \vdash \rho(M_0) : B \supset A}\ \text{IH} \qquad \dfrac{\dfrac{}{\Gamma \vdash \rho(M_1) : B}\ \text{IH} \quad \dfrac{}{\Gamma, x : A \vdash x : A}\ Ax}{\Gamma, y : B \supset A \vdash \langle \underline{y}\rho(M_1)/x\rangle x : A}\ L\supset}{\Gamma \vdash [\rho(M_0)/y]\langle \underline{y}\rho(M_1)/x\rangle x : A}\ Cut$$

and obtain $\Gamma \vdash \langle \{\!\{\rho(M_0)\}\!\}\rho(M_1)/x\rangle x : A$ by Lemma 10 and subject reduction.

2. This follows from the item 3 and Proposition 2.
3. By induction on the derivation of $\Gamma \vdash t : A$, using the property on simply typed λ-terms that $\Gamma \vdash N : B$ and $\Gamma, x : B \vdash M : A$ imply $\Gamma \vdash \{N/x\}M : A$. $\square$