Confluence of Cut-Elimination Procedures for the Intuitionistic Sequent Calculus

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Abstract. We prove confluence of two cut-elimination procedures for the implicational fragment of a standard intuitionistic sequent calculus. One of the cut-elimination procedures uses global proof transformations while the other consists of local ones. Both of them include permutation of cuts to simulate β-reduction in an isomorphic image of the λ-calculus. We establish the confluence properties through a conservativity result on the cut-elimination procedures.
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1 Introduction

Gentzen's cut-elimination theorem [4] has long been a great influence on logic and theoretical computer science. Recent development of structural proof theory is revealing the computational aspect of cut-elimination procedures in the same sense that proof transformations in natural deduction play through the Curry-Howard correspondence [7]. In [8], the author identified a subset of proofs in a standard sequent calculus that correspond to simply typed λ-terms, and defined a reduction relation on those proofs that precisely corresponds to β-reduction of the simply typed λ-calculus. Since the reduction relation is simulated by a local-step cut-elimination procedure, the system of proof terms for the sequent calculus can be considered as a syntactical extension of the λ-calculus including reductions. It is worth noticing that the correspondence holds also for the type-free case, so the reduction system in [8] can simulate the type-free λ-calculus, which means that it is strong enough to represent all computations.

In this paper, we study confluence of a cut-elimination procedure based on the one introduced in [8]. Since the reduction system in [8] is not confluent, we modify one of the reduction rules to a more restricted form. The resulting system is still strong enough to simulate β-reduction in the isomorphic image of the λ-calculus. We also consider another cut-elimination procedure which includes global proof transformations in the style of [2]. The reduction system representing the cut-elimination procedure is similar to one considered in [3], which uses meta-operations like meta-substitution in the λ-calculus.

It is well-known that a local-step cut-elimination procedure has a similarity to explicit substitution calculi. Our proof method is essentially the one often used in the field of explicit substitutions (see, e.g. [1]), called the interpretation method [6]. This method projects reduction steps with explicit substitutions onto those using meta-substitution, and reduces the confluence problem of an explicit substitution calculus to that of the original λ-calculus. To apply this method to the case of a cut-elimination procedure, we need to find an appropriate reduction using meta-operations. Although meta-operations are used in the reduction system for the global cut-elimination procedure mentioned above, it turns out that the system is not appropriate for a target calculus of the method because proving confluence of it has a delicate matter that is not present in the case of the usual λ-calculus. So we define another reduction relation on a certain class of proof terms, and first prove its confluence by the method of parallel reduction [10]. Confluence of the two cut-elimination procedures is inferred from confluence of this reduction by the interpretation method.

Danos et al. [2] proved confluence of their cut-elimination procedures with global proof transformations, depending on confluence of proof nets [5]. In this paper, we give a direct proof of confluence of a similar cut-elimination procedure, using proof terms and meta-operations on them. Our method works also for cut-elimination procedures consisting of local proof transformations and for underlying untyped calculi allowing non-terminating computations.

The paper is organized as follows. In Section 2 we introduce sequent calculus and cut-elimination procedures. In Section 3 we study a subcalculus and meta-operations from the reduction systems. In Section 4 we define another reduction relation and prove its confluence. In Section 5 we prove confluence of the cut-elimination procedures. In Section 6 we conclude by suggestions for future work.

2 Sequent Calculus and Cut-Elimination Procedures

In this section we introduce a term notation for proofs in a standard sequent calculus for intuitionistic implicational logic, following [8]. Our cut-elimination procedures are represented as reduction rules for those terms.

First, the set of raw terms for sequent proofs is defined by the grammar: $t ::= x \mid \lambda x.t \mid \langle xt/x \rangle t \mid [t/x]t$ where x ranges over a denumerable set of variables. $\langle \_\_/\_ \rangle \_$ and $[\_/\_]\_$ are function symbols like explicit substitutions and not meta-substitution ($[\_/\_]\_$ is called the cut-constructor). We use letters x, y, z, w for variables and t, s, r, u for terms. The notions of free and bound variables are defined as usual, with an additional clause that the variable x in $\langle ys/x \rangle t$ or $[s/x]t$ binds the free occurrences of x in t. The set of free variables of a term t is denoted by $FV(t)$. We often use the notation $\langle \underline{x}s/y \rangle t$ to denote $\langle xs/y \rangle t$ if $x \notin FV(s) \cup FV(t)$. The symbol $\equiv$ denotes syntactical equality modulo α-conversion; so for example, $\langle zr/x \rangle \langle \underline{x}s/y \rangle t \equiv \langle zr/w \rangle \langle \underline{w}s/y \rangle t$.

The term assignment for sequent proofs of intuitionistic implicational logic is given in Table 1. We define a context, ranged over by Γ, as a finite set of pairs $\{x_1 : A_1, \ldots, x_n : A_n\}$ where the variables are pairwise distinct. The context $\Gamma, x : A$ denotes the union $\Gamma \cup \{x : A\}$, and $x \notin \Gamma$ means that x does not appear in Γ. For precise representation of proofs by terms, we should specify formulas

Table 1. Sequent calculus and local cut-elimination

$$Ax \frac{}{\Gamma, x : A \vdash x : A} \qquad L\supset \frac{\Gamma \vdash s : A \quad \Gamma, y : B \vdash t : C}{\Gamma, x : A \supset B \vdash \langle xs/y\rangle t : C}\ y \notin \Gamma$$

$$R\supset \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \supset B}\ x \notin \Gamma \qquad Cut \frac{\Gamma \vdash s : A \quad \Gamma, x : A \vdash t : B}{\Gamma \vdash [s/x]t : B}\ x \notin \Gamma$$

$\langle \underline{x}s/y\rangle t$ is used for $\langle xs/y\rangle t$ when $x \notin FV(s) \cup FV(t)$. In that case we assume $x \notin \Gamma$ in the rule $L\supset$.

(1)	$[t/x]y \to y \quad (y \not\equiv x)$
(2)	$[t/x]x \to t$
(3)	$[s/x](\lambda y.t) \to \lambda y.[s/x]t$
(4)	$[r/z]\langle xs/y\rangle t \to \langle x([r/z]s)/y\rangle [r/z]t \quad (x \not\equiv z)$
(5)	$[r/x]\langle xs/y\rangle t \to [r/x]\langle \underline{x}([r/x]s)/y\rangle [r/x]t$ if $x \in FV(s) \cup FV(t)$
(6)	$[z/x]\langle \underline{x}s/y\rangle t \to \langle zs/y\rangle t$
(7′)	$[\langle xs/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to \langle xs/y\rangle [t/z]\langle \underline{z}s'/w\rangle t'$
(*Beta*)	$[\lambda z.r/x]\langle \underline{x}s/y\rangle t \to [[s/z]r/y]t$
(*Perm*$_1$)	$[[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to [r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t'$
(*Perm*$_2$)	$[u/w][\lambda z.r/x]\langle \underline{x}s/y\rangle t \to [[u/w](\lambda z.r)/x][u/w]\langle \underline{x}s/y\rangle t$

on binders, but we will omit them for brevity. If $x \notin FV(s) \cup FV(t)$ in the term $\langle xs/y\rangle t$, we assume $x \notin \Gamma$ in the rule $L\supset$, which means the formula $A \supset B$ is introduced without implicit contraction.

The reduction rules in Table 1 define a cut-elimination procedure consisting of local proof transformations (cf. Appendix A). The reduction relation $\to_{\mathrm{cut}}$ is defined by the contextual closures of these reduction rules. We use $\xrightarrow{+}_{\mathrm{cut}}$ for its transitive closure, and $\xrightarrow{*}_{\mathrm{cut}}$ for its reflexive transitive closure. These kinds of notations are also used for the notions of other reductions in this paper.

The reduction system without the rule (*Beta*) is denoted by $\mathtt{x}$. This subcalculus plays an important role in this paper and is studied in detail in Section 3.

The reduction rules (1) through (5) correspond to cut-elimination steps that permute a cut upwards through its right subproof. The rules (6) and (7′) correspond to steps permuting a cut upwards through its left subproof. The rule (*Beta*) corresponds to the key-case which breaks a cut on an implication into two cuts on its subformulas. The rules (*Perm*$_1$) and (*Perm*$_2$) permute two cuts with some restrictions. In (*Perm*$_1$), the left rule over the lower cut is another cut, and the right rules over both cuts must be $L\supset$ that introduces the cut-formula

Table 2. Global cut-elimination

$(Beta)$	$[\lambda z.r/x]\langle \underline{x}s/y\rangle t$	$\rightarrow$	$[[s/z]r/y]t$	
$(left)$	$[u/x]\langle \underline{x}s/y\rangle t$	$\rightarrow$	$\langle \{\!\{u\}\!\}s/y\rangle t$	if u is not of the form $\lambda z.r$
$(right)$	$[u/x]r$	$\rightarrow$	$\{u/x\}r$	if r is not of the form $\langle \underline{x}s/y\rangle t$

where $\{\_/\_\}\_$ and $\langle \{\!\{\_\}\!\}\_/\_\rangle\_$ are the meta-operations defined as follows:

$$\begin{aligned}
\{u/x\}y &=_{def} y \qquad (y \not\equiv x)\\
\{u/x\}x &=_{def} u\\
\{u/x\}(\lambda y.t) &=_{def} \lambda y.\{u/x\}t\\
\{u/x\}\langle zs/y\rangle t &=_{def} \langle z(\{u/x\}s)/y\rangle\{u/x\}t \qquad (z \not\equiv x)\\
\{u/x\}\langle xs/y\rangle t &=_{def} [u/x]\langle \underline{x}(\{u/x\}s)/y\rangle\{u/x\}t\\
\{u/x\}[s/y]t &=_{def} [\{u/x\}s/y]\{u/x\}t
\end{aligned}$$

$$\begin{aligned}
\langle \{\!\{z\}\!\}s/y\rangle t &=_{def} \langle zs/y\rangle t\\
\langle \{\!\{\lambda z.r\}\!\}s/y\rangle t &=_{def} [\lambda z.r/x]\langle \underline{x}s/y\rangle t\\
\langle \{\!\{\langle zs'/w\rangle r\}\!\}s/y\rangle t &=_{def} \langle zs'/w\rangle\langle \{\!\{r\}\!\}s/y\rangle t\\
\langle \{\!\{[s'/w]r\}\!\}s/y\rangle t &=_{def} [s'/w]\langle \{\!\{r\}\!\}s/y\rangle t
\end{aligned}$$

without implicit contraction. In $(Perm_2)$, the right rule over the lower cut is another cut, which must construct a proof corresponding to a redex of the rule $(Beta)$.

The original cut-elimination procedure in [8] uses the following rule (7) instead of (7′):

$$(7) \quad [\langle xs/y\rangle t/z]r \rightarrow \langle xs/y\rangle[t/z]r$$

This rule makes the cut-elimination procedure non-confluent (e.g., the critical pair $w \leftarrow [\langle xs/y\rangle t/z]w \rightarrow \langle xs/y\rangle[t/z]w$ is not joinable). For a confluent cut-elimination procedure, it is therefore necessary to restrict reductions. The rule (7′) restricts the rule (7) so that the right rule over the cut must be $L \supset$ that introduces the cut-formula without implicit contraction. As shown in [8], this cut-elimination procedure is still strong enough to simulate β-reduction in the isomorphic image of the λ-calculus.

Table 2 presents another cut-elimination procedure which includes global proof transformations. The cut-elimination procedure is implemented by reduction rules that use meta-operations $\{\_/\_\}\_$ and $\langle \{\!\{\_\}\!\}\_/\_\rangle\_$, analogously to proof transformations in natural deduction. The operation $\langle \{\!\{\_\}\!\}\_/\_\rangle\_$ corresponds to the cut-elimination process where the right rule over the cut is $L \supset$ introducing the cut-formula without implicit contraction, and the cut is permuted upwards through its left subproof. Note that the conditions of (*left*) and (*right*) make the cut-elimination procedure first permute a cut upwards through its right subproof and then through its left subproof. The reduction relation generated by the rules (*Beta*), (*left*) and (*right*) is denoted by $\to_{\mathrm{gcut}}$.

The following lemma is immediate from the definition of $\{\_/\_\}\_$.

Lemma 1. *If $x \notin FV(t)$ then $\{u/x\}t \equiv t$.*

Proof. By induction on the structure of t. □

3 The Subcalculus x and Meta-Operations

In this section we study properties of the subcalculus $\mathtt{x}$ which is the reduction system in Table 1 without the rule (*Beta*). In the typed case, it corresponds to the cut-elimination steps except the key-case, i.e., the case where both left and right rules over the cut rule introduce the cut-formula. We show that the subcalculus $\mathtt{x}$ is strongly normalizing and confluent, and investigate its relation to the meta-operations in Table 2.

First we give a technical definition to prove strong normalization of the subcalculus $\mathtt{x}$.

Definition 1. *A term $[s/x]t$ is called an* application term *if t is one of the forms: $[u/w]\langle \underline{x}s'/y\rangle t'$, $\langle \underline{x}s'/y\rangle t'$ and $[\langle \underline{x}s'/y\rangle t'/z]\langle \underline{z}s''/w\rangle t''$, where x occurs only once in t.*

Lemma 2. *If $[s/x]t$ is an application term and $t \to_{\mathtt{x}} t'$, then $[s/x]t'$ is also an application term.*

Proof. It suffices to check each case. □

Proposition 1. *The subcalculus $\mathtt{x}$ is strongly normalizing.*

Proof. The proof is by interpretation. We define a function h as follows:

$$
\begin{array}{ll}
h(x) & =_{def} 1 \\
h(\lambda x.t) & =_{def} h(t)+1 \\
h(\langle xs/y\rangle t) & =_{def} h(s)+h(t)+1 \\
h([s/x]t) & =_{def} \begin{cases} (h(s)+1)^2 \times h(t) & \text{if } [s/x]t \text{ is an application term} \\ (h(s)+1)^{2\times h(t)} & \text{otherwise} \end{cases}
\end{array}
$$

and observe that if $t \to_{\mathtt{x}} t'$ then $h(t) > h(t')$. If $t \equiv [s/x]r$ is an application term and $r \to_{\mathtt{x}} r'$, then we use Lemma 2. □

Proposition 2. *The subcalculus $\mathtt{x}$ is confluent.*

Proof. By Newman's Lemma, it suffices to check the local confluence. There are two critical pairs caused by the rules $(7')$ and $(Perm_1)$, and by $(Perm_1)$ and $(Perm_1)$, both of which are joinable. □

As a result, we can define the unique $\mathtt{x}$-normal form of each term.

Definition 2. *The unique* $\mathtt{x}$*-normal form of a term* t *is denoted by* $\mathtt{x}(t)$*.*

A term in which every cut-constructor forms a redex of the rule (*Beta*) is called a *Beta-term*. The relation between *Beta*-terms and $\mathtt{x}$-normal forms is as follows.

Proposition 3. t *is a Beta-term if and only if* t *is in* $\mathtt{x}$*-normal form.*

Proof. The only if part is by induction on the structure of *Beta*-terms. We prove the if part by induction on the structure of t. Suppose that t is in $\mathtt{x}$-normal form. Then by the induction hypothesis, all subterms of t are *Beta*-terms. Now, if t is not a *Beta*-term then t is of the form $[u/x]r(\not\equiv [\lambda z.r'/x]\langle \underline{x}s/y\rangle t')$ where u, r are *Beta*-terms. In this case, t is an $\mathtt{x}$-redex, which is a contradiction. □

The next lemma shows that the subcalculus $\mathtt{x}$ correctly simulates the meta-operations on *Beta*-terms.

Lemma 3. *Let* u, t, s *be Beta-terms. Then*

1. $[u/x]t \xrightarrow{*}_{\mathtt{x}} \{u/x\}t$,
2. $[u/x]\langle \underline{x}s/y\rangle t \xrightarrow{*}_{\mathtt{x}} \langle \{\!\{u\}\!\}s/y\rangle t$*. Moreover,* $\langle \{\!\{u\}\!\}s/y\rangle t$ *is a Beta-term, hence* $\mathtt{x}([u/x]\langle \underline{x}s/y\rangle t) \equiv \langle \{\!\{u\}\!\}s/y\rangle t$*.*

Proof. 1. By induction on the structure of t.
2. By induction on the structure of u. □

Next we show that $\to_{\mathrm{gcut}}$ is sufficient to reach $\mathtt{x}$-normal forms.

Lemma 4. *Let* u, s, t *be Beta-terms. Then*

1. $[u/x]\langle \underline{x}s/y\rangle t \xrightarrow{*}_{\mathrm{gcut}} \mathtt{x}([u/x]\langle \underline{x}s/y\rangle t)$,
2. $\{u/x\}t \xrightarrow{*}_{\mathrm{gcut}} \mathtt{x}(\{u/x\}t)$.

Proof. 1. If $u \equiv \lambda z.r$ then $[u/x]\langle \underline{x}s/y\rangle t \equiv \mathtt{x}([u/x]\langle \underline{x}s/y\rangle t)$. If u is not of the form $\lambda z.r$, then $[u/x]\langle \underline{x}s/y\rangle t' \to_{left} \langle \{\!\{u\}\!\}s/y\rangle t' \equiv \mathtt{x}([u/x]\langle \underline{x}s/y\rangle t')$ by Lemma 3 (2).
2. By induction on the structure of t. □

Lemma 5. $t \xrightarrow{*}_{\mathrm{gcut}} \mathtt{x}(t)$*.*

Proof. By induction on the structure of t. □

The following lemmas are essential to the parallel reduction method in the next section. Note that $\{u/x\}\langle \{\!\{t\}\!\}s/y\rangle t' \equiv \langle \{\!\{\{u/x\}t\}\!\}s/y\rangle t'$ instead of Lemma 7 does not hold in general; for example, $\{z/x\}\langle \{\!\{x\}\!\}w/y\rangle w' \equiv [z/x]\langle xw/y\rangle w' \not\equiv \langle zw/y\rangle w' \equiv \langle \{\!\{\{z/x\}x\}\!\}w/y\rangle w'$. This makes it difficult to apply a direct parallel reduction method to $\to_{\mathrm{gcut}}$. So we consider the meta-operation $\{\_/\_\}\_$ followed by $\mathtt{x}$-reductions to $\mathtt{x}$-normal forms (i.e., $\mathtt{x}(\{\_/\_\}\_)$), and in the next section we define another reduction relation that matches such operation.

Lemma 6. $\langle \{\!\{ \langle \{\!\{ u \}\!\} s/y \rangle t \}\!\} s'/y' \rangle t' \equiv \langle \{\!\{ u \}\!\} s/y \rangle \langle \{\!\{ t \}\!\} s'/y' \rangle t'$.

Proof. By induction on the structure of u. □

Lemma 7. *Let u, t, s, t' be Beta-terms. Then*
$\mathtt{x}(\{u/x\}\langle \{\!\{ t \}\!\} s/y \rangle t') \equiv \langle \{\!\{ \mathtt{x}(\{u/x\}t) \}\!\} \mathtt{x}(\{u/x\}s)/y \rangle \mathtt{x}(\{u/x\}t')$.
In particular, if $x \notin FV(s) \cup FV(t')$ then
$\mathtt{x}(\{u/x\}\langle \{\!\{ t \}\!\} s/y \rangle t') \equiv \langle \{\!\{ \mathtt{x}(\{u/x\}t) \}\!\} s/y \rangle t'$.

Proof. By induction on the structure of t. □

Lemma 8. *Let u, s, t be Beta-terms with $y \notin FV(u)$. Then*
$\mathtt{x}(\{u/x\}\mathtt{x}(\{s/y\}t)) \equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}\mathtt{x}(\{u/x\}t))$.

Proof. By induction on the structure of t. □

4 Confluence of β-Reduction

In this section we introduce another reduction relation on *Beta*-terms and show that it is confluent by the parallel reduction method [10]. Confluence of the two cut-elimination procedures is proved using projections onto this reduction.

The reduction relation $\to_\beta$ on *Beta*-terms is defined by the contextual closure of the rule:

$$(\beta) \qquad [\lambda z.r/x]\langle \underline{x} s/y \rangle t \to \mathtt{x}(\{\mathtt{x}(\{s/z\}r)/y\}t)$$

This reduction relation is indeed an extension of β-reduction on pure terms (i.e., the isomorphic image of λ-terms) in [8].

Proposition 4. *Let t, t' be Beta-terms.*

1. *If $t \to_\beta t'$ then $t \xrightarrow{+}_{\text{cut}} t'$.*
2. *If $t \to_\beta t'$ then $t \xrightarrow{+}_{\text{gcut}} t'$.*

Proof. By induction on the reduction relation $\to_\beta$. We treat the case where the reduction is at the root. Then

$$\begin{aligned}
[\lambda z.r/x]\langle \underline{x} s/y \rangle t_0 &\to_{Beta} [[s/z]r/y]t_0 \\
&\xrightarrow{*}_{\mathtt{x}} \mathtt{x}([\mathtt{x}([s/z]r)/y]t_0) && (*) \\
&\equiv \mathtt{x}([\mathtt{x}(\{s/z\}r)/y]t_0) && \text{(by Lemma 3 (1))} \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{s/z\}r)/y\}t_0) && \text{(by Lemma 3 (1))}
\end{aligned}$$

where the step $(*)$ can also be established with $\xrightarrow{*}_{\text{gcut}}$ by Lemma 5. □

The parallel reduction $\Rightarrow$ for $\to_\beta$ is defined by the rules in Table 3.

Lemma 9. *For every Beta-term t, $t \Rightarrow t$.*

Table 3. Parallel reduction

$$\frac{}{x \Rightarrow x}\ (pr_1) \qquad \frac{t \Rightarrow t'}{\lambda x.t \Rightarrow \lambda x.t'}\ (pr_2) \qquad \frac{s \Rightarrow s' \quad t \Rightarrow t'}{\langle xs/y\rangle t \Rightarrow \langle xs'/y\rangle t'}\ (pr_3)$$

$$\frac{r \Rightarrow r' \quad s \Rightarrow s' \quad t \Rightarrow t'}{[\lambda z.r/x]\langle \underline{x}s/y\rangle t \Rightarrow [\lambda z.r'/x]\langle \underline{x}s'/y\rangle t'}\ (pr_4)$$

$$\frac{r \Rightarrow r' \quad s \Rightarrow s' \quad t \Rightarrow t'}{[\lambda z.r/x]\langle \underline{x}s/y\rangle t \Rightarrow \mathtt{x}(\{\mathtt{x}(\{s'/z\}r')/y\}t')}\ (pr_5)$$

Proof. By induction on the structure of t. □

Lemma 10.

1. *If* $t \to_\beta t'$ *then* $t \Rightarrow t'$.
2. *If* $t \Rightarrow t'$ *then* $t \xrightarrow{*}_\beta t'$.
3. *If* $u \Rightarrow u'$, $s \Rightarrow s'$ *and* $t \Rightarrow t'$ *then* $\langle \{\!\!\{u\}\!\!\}s/y\rangle t \Rightarrow \langle \{\!\!\{u'\}\!\!\}s'/y\rangle t'$.
4. *If* $u \Rightarrow u'$ *and* $t \Rightarrow t'$ *then* $\mathtt{x}(\{u/x\}t) \Rightarrow \mathtt{x}(\{u'/x\}t')$.

Proof. 1. By induction on the reduction relation $\to_\beta$.
2. By induction on the definition of $t \Rightarrow t'$.
3. By induction on the definition of $u \Rightarrow u'$.
4. By induction on the definition of $t \Rightarrow t'$. □

Definition 3. *For each Beta-term* t, *the term* $t^\star$ *is defined inductively as follows:*

1. $x^\star =_{def} x$,
2. $(\lambda x.t)^\star =_{def} \lambda x.t^\star$,
3. $(\langle xs/y\rangle t)^\star =_{def} \langle xs^\star/y\rangle t^\star$,
4. $([\lambda z.r/x]\langle \underline{x}s/y\rangle t)^\star =_{def} \mathtt{x}(\{\mathtt{x}(\{s^\star/z\}r^\star)/y\}t^\star)$.

Lemma 11. *If* $t \Rightarrow t'$ *then* $t' \Rightarrow t^\star$.

Proof. By induction on the definition of $t \Rightarrow t'$. □

Lemma 12. *If* $t \Rightarrow t_1$ *and* $t \Rightarrow t_2$ *then there is* t' *such that* $t_1 \Rightarrow t'$ *and* $t_2 \Rightarrow t'$.

Proof. By Lemma 11. □

Theorem 1. *The reduction relation* $\to_\beta$ *is confluent.*

Proof. By Lemmas 10 and 12. □

Lemma 13. *Let* u, t *be Beta-terms.*

1. *If* $u \to_\beta u'$ *then* $\mathtt{x}(\{u/x\}t) \xrightarrow{*}_\beta \mathtt{x}(\{u'/x\}t)$.
2. *If* $t \to_\beta t'$ *then* $\mathtt{x}(\{u/x\}t) \xrightarrow{*}_\beta \mathtt{x}(\{u/x\}t')$.

Proof. These are derived from Lemmas 9 and 10. □

5 Confluence of Cut-Elimination Procedures

In this section we complete the proofs of confluence of the cut-elimination procedures. We also establish a conservativity result among the cut-elimination procedures and β-reduction on *Beta*-terms.

Lemma 14.

1. $\mathtt{x}(\langle \{\!\!\{u\}\!\!\} s/y\rangle t) \equiv \langle \{\!\!\{\mathtt{x}(u)\}\!\!\} \mathtt{x}(s)/y\rangle \mathtt{x}(t)$,
2. $\mathtt{x}(\{u/x\}t) \equiv \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(t))$.

Proof. 1. By induction on the structure of u.
2. By induction on the structure of t. □

The next two lemmas show that the cut-elimination procedures project onto β-reduction on *Beta*-terms.

Lemma 15. *If* $t \to_{\mathrm{gcut}} t'$ *then* $\mathtt{x}(t) \xrightarrow{*}_{\beta} \mathtt{x}(t')$.

Proof. By induction on the reduction relation $\to_{\mathrm{gcut}}$. □

Lemma 16. *If* $t \to_{\mathrm{cut}} t'$ *then* $\mathtt{x}(t) \xrightarrow{*}_{\beta} \mathtt{x}(t')$.

Proof. If $t \to_{\mathtt{x}} t'$ then $\mathtt{x}(t) \equiv \mathtt{x}(t')$. So it suffices to show that if $t \to_{Beta} t'$ then $\mathtt{x}(t) \xrightarrow{*}_{\beta} \mathtt{x}(t')$. This is proved in a similar way to Lemma 15. □

Now we have a conservativity result among the reductions on *Beta*-terms.

Theorem 2. *For any Beta-terms* t, t'*, the following are equivalent.*

1. $t \xrightarrow{*}_{\mathrm{gcut}} t'$
2. $t \xrightarrow{*}_{\mathrm{cut}} t'$
3. $t \xrightarrow{*}_{\beta} t'$

Proof. By Lemmas 15 and 16, and Proposition 4. □

We are now ready to show that the reduction relations $\to_{\mathrm{gcut}}$ and $\to_{\mathrm{cut}}$ are confluent, using confluence of $\to_{\beta}$ on *Beta*-terms (Theorem 1). The results also hold in the typed case, so that confluence of the cut-elimination procedures follows.

Theorem 3.

1. *The reduction relation* $\to_{\mathrm{gcut}}$ *is confluent.*
2. *The reduction relation* $\to_{\mathrm{cut}}$ *is confluent.*

Proof. 1. Suppose that $t \xrightarrow{*}_{\mathrm{gcut}} t_1$ and $t \xrightarrow{*}_{\mathrm{gcut}} t_2$. Then by Lemma 15, $\mathtt{x}(t) \xrightarrow{*}_{\beta} \mathtt{x}(t_i)$ $(i = 1, 2)$, so by confluence of $\to_{\beta}$, there is a *Beta*-term t' such that $\mathtt{x}(t_i) \xrightarrow{*}_{\beta} t'$ $(i = 1, 2)$. Since $\mathtt{x}(t_i) \xrightarrow{*}_{\mathrm{gcut}} t'$ by Theorem 2 and $t_i \xrightarrow{*}_{\mathrm{gcut}} \mathtt{x}(t_i)$ by Lemma 5, we have $t_i \xrightarrow{*}_{\mathrm{gcut}} t'$ $(i = 1, 2)$.
2. Similar, using Lemma 16 instead of Lemma 15. □

6 Conclusion

We have proved confluence of global and local cut-elimination procedures, using proof terms for a standard sequent calculus of intuitionistic logic. For the interpretation method to work, we have introduced β-reduction on *Beta*-terms, and proved its confluence by the method of parallel reduction. Then confluence of the two cut-elimination procedures has been obtained through projections onto the β-reduction. Additionally, we have established a conservativity result among the cut-elimination procedures and the β-reduction. Note that our proofs are also effective in the type-free case allowing non-terminating computations.

The problem on substitution lemmas (cf. the remark before Lemma 6) was also pointed out in [11, page 136] for the case of the classical sequent calculus. In future work, we will investigate the relation between their observations and ours, and develop proofs of confluence for some cut-elimination procedures in the classical sequent calculus.

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A Cut-Elimination Steps for Sequent Proofs

In this appendix we display the cut-elimination steps corresponding to the reduction rules in Table 1. First we need the following lemma.

Lemma 17. *Let $\Gamma \vdash t : A$ where x does not appear in Γ. Then $\Gamma, x : B \vdash t : A$.*

Proof. By induction on the structure of derivations. □

In the following $\colorbox{lightgray}{$x : B$}$ means a declaration added by the above lemma.

Cut-Elimination Steps for Sequent Proofs

(1) $[t/x]y \to y \quad (y \not\equiv x)$

$$\dfrac{\Gamma, y : A \vdash t : B \quad \dfrac{}{\Gamma, x : B, y : A \vdash y : A}\,Ax}{\Gamma, y : A \vdash [t/x]y : A}\,Cut \quad \to \quad \dfrac{}{\Gamma, y : A \vdash y : A}\,Ax$$

(2) $[t/x]x \to t$

$$\dfrac{\Gamma \vdash t : A \quad \dfrac{}{\Gamma, x : A \vdash x : A}\,Ax}{\Gamma \vdash [t/x]x : A}\,Cut \quad \to \quad \Gamma \vdash t : A$$

(3) $[s/x](\lambda y.t) \to \lambda y.[s/x]t$

$$\dfrac{\Gamma \vdash s : A \quad \dfrac{\Gamma, x : A, y : B \vdash t : C}{\Gamma, x : A \vdash \lambda y.t : B \supset C}\,R\supset}{\Gamma \vdash [s/x](\lambda y.t) : B \supset C}\,Cut \quad \to \quad \dfrac{\dfrac{\Gamma, \colorbox{lightgray}{$y : B$} \vdash s : A \quad \Gamma, x : A, y : B \vdash t : C}{\Gamma, y : B \vdash [s/x]t : C}\,Cut}{\Gamma \vdash \lambda y.[s/x]t : B \supset C}\,R\supset$$

(4) $[r/z]\langle xs/y\rangle t \to \langle x([r/z]s)/y\rangle [r/z]t \quad (x \not\equiv z)$

$$\dfrac{\Gamma, x : B \supset C \vdash r : A \quad \dfrac{\Gamma, z : A \vdash s : B \quad \Gamma, z : A, y : C \vdash t : D}{\Gamma, z : A, x : B \supset C \vdash \langle xs/y\rangle t : D}\,L\supset}{\Gamma, x : B \supset C \vdash [r/z]\langle xs/y\rangle t : D}\,Cut$$

$$\to \dfrac{\dfrac{\Gamma, x : B \supset C \vdash r : A \quad \Gamma, z : A, \colorbox{lightgray}{$x : B \supset C$} \vdash s : B}{\Gamma, x : B \supset C \vdash [r/z]s : B}\,Cut \quad \dfrac{\Gamma, x : B \supset C, \colorbox{lightgray}{$y : C$} \vdash r : A \quad \Gamma, z : A, \colorbox{lightgray}{$x : B \supset C$}, y : C \vdash t : D}{\Gamma, x : B \supset C, y : C \vdash [r/z]t : D}\,Cut}{\Gamma, x : B \supset C \vdash \langle x([r/z]s)/y\rangle [r/z]t : D}\,L\supset$$

(5) $[r/x]\langle xs/y\rangle t \to [r/x]\langle \underline{x}([r/x]s)/y\rangle [r/x]t \quad$ if $x \in FV(s) \cup FV(t)$

$$\dfrac{\Gamma \vdash r : A \supset B \quad \dfrac{\Gamma, x : A \supset B \vdash s : A \quad \Gamma, x : A \supset B, y : B \vdash t : C}{\Gamma, x : A \supset B \vdash \langle xs/y\rangle t : C}\,L\supset}{\Gamma \vdash [r/x]\langle xs/y\rangle t : C}\,Cut$$

$$\to \dfrac{\Gamma \vdash r : A \supset B \quad \dfrac{\dfrac{\Gamma \vdash r : A \supset B \quad \Gamma, x : A \supset B \vdash s : A}{\Gamma \vdash [r/x]s : A}\,Cut \quad \dfrac{\Gamma, \colorbox{lightgray}{$y : B$} \vdash r : A \supset B \quad \Gamma, x : A \supset B, y : B \vdash t : C}{\Gamma, y : B \vdash [r/x]t : C}\,Cut}{\Gamma, x : A \supset B \vdash \langle \underline{x}([r/x]s)/y\rangle [r/x]t : C}\,L\supset}{\Gamma \vdash [r/x]\langle \underline{x}([r/x]s)/y\rangle [r/x]t : C}\,Cut$$

(6) $[z/x]\langle \underline{x}s/y\rangle t \to \langle zs/y\rangle t$

$$\dfrac{\dfrac{}{\Gamma, z : A \supset B \vdash z : A \supset B}\,\mathit{Ax} \qquad \dfrac{\Gamma, z : A \supset B \vdash s : A \quad \Gamma, z : A \supset B, y : B \vdash t : C}{\Gamma, z : A \supset B, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C}\,L\supset}{\Gamma, z : A \supset B \vdash [z/x]\langle \underline{x}s/y\rangle t : C}\,\mathit{Cut}$$

$$\to \quad \dfrac{\Gamma, z : A \supset B \vdash s : A \quad \Gamma, z : A \supset B, y : B \vdash t : C}{\Gamma, z : A \supset B \vdash \langle zs/y\rangle t : C}\,L\supset$$

(7′) $[\langle xs/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to \langle xs/y\rangle [t/z]\langle \underline{z}s'/w\rangle t'$

$$\dfrac{\dfrac{\Gamma \vdash s : A \quad \Gamma, y : B \vdash t : C \supset D}{\Gamma, x : A \supset B \vdash \langle xs/y\rangle t : C \supset D}\,L\supset \qquad \Gamma, x : A \supset B, z : C \supset D \vdash \langle \underline{z}s'/w\rangle t' : E}{\Gamma, x : A \supset B \vdash [\langle xs/y\rangle t/z]\langle \underline{z}s'/w\rangle t' : E}\,\mathit{Cut}$$

$$\to \quad \dfrac{\Gamma, x : A \supset B \vdash s : A \qquad \dfrac{\Gamma, x : A \supset B, y : B \vdash t : C \supset D \quad \Gamma, x : A \supset B, y : B, z : C \supset D \vdash \langle \underline{z}s'/w\rangle t' : E}{\Gamma, x : A \supset B, y : B \vdash [t/z]\langle \underline{z}s'/w\rangle t' : E}\,\mathit{Cut}}{\Gamma, x : A \supset B \vdash \langle xs/y\rangle [t/z]\langle \underline{z}s'/w\rangle t' : E}\,L\supset$$

(Beta) $[\lambda z.r/x]\langle \underline{x}s/y\rangle t \to [[s/z]r/y]t$

$$\dfrac{\dfrac{\Gamma, z : A \vdash r : B}{\Gamma \vdash \lambda z.r : A \supset B}\,R\supset \qquad \dfrac{\Gamma \vdash s : A \quad \Gamma, y : B \vdash t : C}{\Gamma, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C}\,L\supset}{\Gamma \vdash [\lambda z.r/x]\langle \underline{x}s/y\rangle t : C}\,\mathit{Cut}$$

$$\to \quad \dfrac{\dfrac{\Gamma \vdash s : A \quad \Gamma, z : A \vdash r : B}{\Gamma \vdash [s/z]r : B}\,\mathit{Cut} \qquad \Gamma, y : B \vdash t : C}{\Gamma \vdash [[s/z]r/y]t : C}\,\mathit{Cut}$$

(Perm_1) $[[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to [r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t'$

$$\dfrac{\dfrac{\Gamma \vdash r : A \supset B \quad \Gamma, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C \supset D}{\Gamma \vdash [r/x]\langle \underline{x}s/y\rangle t : C \supset D}\,\mathit{Cut} \qquad \Gamma, z : C \supset D \vdash \langle \underline{z}s'/w\rangle t' : E}{\Gamma \vdash [[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' : E}\,\mathit{Cut}$$

$$\to \quad \dfrac{\Gamma \vdash r : A \supset B \qquad \dfrac{\Gamma, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C \supset D \quad \Gamma, x : A \supset B, z : C \supset D \vdash \langle \underline{z}s'/w\rangle t' : E}{\Gamma, x : A \supset B \vdash [\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' : E}\,\mathit{Cut}}{\Gamma \vdash [r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' : E}\,\mathit{Cut}$$

(Perm_2) $[u/w][\lambda z.r/x]\langle \underline{x}s/y\rangle t \to [[u/w](\lambda z.r)/x][u/w]\langle \underline{x}s/y\rangle t$

$$\dfrac{\Gamma \vdash u : D \qquad \dfrac{\Gamma, w : D \vdash \lambda z.r : A \supset B \quad \Gamma, w : D, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C}{\Gamma, w : D \vdash [\lambda z.r/x]\langle \underline{x}s/y\rangle t : C}\,\mathit{Cut}}{\Gamma \vdash [u/w][\lambda z.r/x]\langle \underline{x}s/y\rangle t : C}\,\mathit{Cut}$$

$$\to \quad \dfrac{\dfrac{\Gamma \vdash u : D \quad \Gamma, w : D \vdash \lambda z.r : A \supset B}{\Gamma \vdash [u/w](\lambda z.r) : A \supset B}\,\mathit{Cut} \qquad \dfrac{\Gamma, x : A \supset B \vdash u : D \quad \Gamma, w : D, x : A \supset B \vdash \langle \underline{x}s/y\rangle t : C}{\Gamma, x : A \supset B \vdash [u/w]\langle \underline{x}s/y\rangle t : C}\,\mathit{Cut}}{\Gamma \vdash [[u/w](\lambda z.r)/x][u/w]\langle \underline{x}s/y\rangle t : C}\,\mathit{Cut}$$

B Proofs in Section 3

In this appendix we give proofs of some propositions and lemmas in Section 3.

Proposition 1. *The subcalculus* $\mathtt{x}$ *is strongly normalizing.*

Proof. The proof is by interpretation. We define a function h as follows:

$$
\begin{array}{ll}
h(x) & =_{def} 1 \\
h(\lambda x.t) & =_{def} h(t)+1 \\
h(\langle xs/y\rangle t) & =_{def} h(s)+h(t)+1 \\
h([s/x]t) & =_{def} \begin{cases} (h(s)+1)^2 \times h(t) & \text{if } [s/x]t \text{ is an application term} \\ (h(s)+1)^{2\times h(t)} & \text{otherwise} \end{cases}
\end{array}
$$

and show by induction on the structure of t that if $t \to_{\mathtt{x}} t'$ then $h(t) > h(t')$. First we consider the case where the reduction is at the root.

(1) $[t/x]y \to y \quad (y \not\equiv x)$

$$
\begin{array}{lll}
\text{LHS}: h([t/x]y) & = (h(t)+1)^{2\times h(y)} \\
& = (h(t)+1)^2 \\
\text{RHS}: h(y) & = 1
\end{array}
$$

(2) $[t/x]x \to t$

$$
\begin{array}{lll}
\text{LHS}: h([t/x]x) & = (h(t)+1)^{2\times h(x)} \\
& = (h(t)+1)^2 \\
\text{RHS}: h(t) & = h(t)
\end{array}
$$

(3) $[s/x](\lambda y.t) \to \lambda y.[s/x]t$

$$
\begin{array}{ll}
\text{LHS}: h([s/x](\lambda y.t)) & = (h(s)+1)^{2\times h(\lambda y.t)} \\
& = (h(s)+1)^{2\times (h(t)+1)} \\
\text{RHS}: h(\lambda y.[s/x]t) & = h([s/x]t)+1 \\
& = (h(s)+1)^{2\times h(t)}+1
\end{array}
$$

If $[s/x]t$ is an application term then the RHS is even smaller.

(4) $[r/z]\langle xs/y\rangle t \to \langle x([r/z]s)/y\rangle [r/z]t \quad (x \not\equiv z)$

$$
\begin{array}{ll}
\text{LHS}: h([r/z]\langle xs/y\rangle t) & = (h(r)+1)^{2\times h(\langle xs/y\rangle t)} \\
& = (h(r)+1)^{2\times (h(s)+h(t)+1)} \\
\text{RHS}: h(\langle x([r/z]s)/y\rangle [r/z]t) & = h([r/z]s)+h([r/z]t)+1 \\
& = (h(r)+1)^{2\times h(s)}+(h(r)+1)^{2\times h(t)}+1
\end{array}
$$

If $[r/z]s$ or $[r/z]t$ is an application term then the RHS is even smaller.

(5) $[r/x]\langle xs/y\rangle t \to [r/x]\langle \underline{x}([r/x]s)/y\rangle[r/x]t \quad \text{if } x \in FV(s) \cup FV(t)$

$$\begin{aligned}
\text{LHS}: h([r/x]\langle xs/y\rangle t) &= (h(r)+1)^{2\times h(\langle xs/y\rangle t)}\\
&= (h(r)+1)^{2\times(h(s)+h(t)+1)}\\
&= (h(r)+1)^2 \times (h(r)+1)^{2\times(h(s)+h(t))}\\
\text{RHS}: h([r/x]\langle \underline{x}([r/x]s)/y\rangle[r/x]t) &= (h(r)+1)^2 \times h(\langle \underline{x}([r/x]s)/y\rangle[r/x]t)\\
&= (h(r)+1)^2 \times (h([r/x]s) + h([r/x]t)+1)\\
&= (h(r)+1)^2 \times ((h(r)+1)^{2\times h(s)} + (h(r)+1)^{2\times h(t)}+1)
\end{aligned}$$

If $[r/x]s$ or $[r/x]t$ is an application term then the RHS is even smaller.

(6) $[z/x]\langle \underline{x}s/y\rangle t \to \langle zs/y\rangle t$

$$\begin{aligned}
\text{LHS}: h([z/x]\langle \underline{x}s/y\rangle t) &= (h(z)+1)^2 \times h(\langle \underline{x}s/y\rangle t)\\
&= (1+1)^2 \times (h(s)+h(t)+1)\\
\text{RHS}: h(\langle zs/y\rangle t) &= h(s)+h(t)+1
\end{aligned}$$

(7′) $[\langle xs/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to \langle xs/y\rangle[t/z]\langle \underline{z}s'/w\rangle t'$

$$\begin{aligned}
\text{LHS}: h([\langle xs/y\rangle t/z]\langle \underline{z}s'/w\rangle t') &= (h(\langle xs/y\rangle t)+1)^2 \times h(\langle \underline{z}s'/w\rangle t')\\
&= ((h(s)+h(t)+1)+1)^2 \times h(\langle \underline{z}s'/w\rangle t')\\
\text{RHS}: h(\langle xs/y\rangle[t/z]\langle \underline{z}s'/w\rangle t') &= h(s) + h([t/z]\langle \underline{z}s'/w\rangle t') + 1\\
&= h(s) + (h(t)+1)^2 \times h(\langle \underline{z}s'/w\rangle t') + 1
\end{aligned}$$

$(Perm_1)$ $[[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t' \to [r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t'$

$$\begin{aligned}
\text{LHS}: h([[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t') &= (h([r/x]\langle \underline{x}s/y\rangle t)+1)^2 \times h(\langle \underline{z}s'/w\rangle t')\\
&= ((h(r)+1)^2 \times h(\langle \underline{x}s/y\rangle t)+1)^2 \times h(\langle \underline{z}s'/w\rangle t')\\
\text{RHS}: h([r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t') &= (h(r)+1)^2 \times h([\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t')\\
&= (h(r)+1)^2 \times (h(\langle \underline{x}s/y\rangle t)+1)^2 \times h(\langle \underline{z}s'/w\rangle t')
\end{aligned}$$

$(Perm_2)$ $[u/w][\lambda z.r/x]\langle \underline{x}s/y\rangle t \to [[u/w](\lambda z.r)/x][u/w]\langle \underline{x}s/y\rangle t$

$$\begin{aligned}
\text{LHS}: h([u/w][\lambda z.r/x]\langle \underline{x}s/y\rangle t) &= (h(u)+1)^{2\times h([\lambda z.r/x]\langle \underline{x}s/y\rangle t)}\\
&= (h(u)+1)^{2\times(h(\lambda z.r)+1)^2\times h(\langle \underline{x}s/y\rangle t)}\\
\text{RHS}: h([[u/w](\lambda z.r)/x][u/w]\langle \underline{x}s/y\rangle t) &= (h([u/w](\lambda z.r))+1)^2 \times h([u/w]\langle \underline{x}s/y\rangle t)\\
&= ((h(u)+1)^{2\times h(\lambda z.r)}+1)^2 \times (h(u)+1)^{2\times h(\langle \underline{x}s/y\rangle t)}
\end{aligned}$$

The case where the reduction is not at the root is immediate form the induction hypothesis. Note that if $t \equiv [s/x]r$ is an application term and $r \to_{\mathtt{x}} r'$, then $t \equiv [s/x]r'$ is an application term by Lemma 2. □

Proposition 2. *The subcalculus* $\mathtt{x}$ *is confluent.*

Proof. By Newman's Lemma, it suffices to check the local confluence. We consider the following two cases where the LHS of a rule unifies with a non-variable subterm of the LHS of another rule.

(a) $[[\langle xs/y\rangle t/z]\langle \underline{z}s'/w\rangle t'/z']\langle \underline{z'}s''/w'\rangle t''$ is reduced to
$[\langle xs/y\rangle [t/z]\langle \underline{z}s'/w\rangle t'/z']\langle \underline{z'}s''/w'\rangle t''$ by the rule $(7')$, or to
$[\langle xs/y\rangle t/z][\langle \underline{z}s'/w\rangle t'/z']\langle \underline{z'}s''/w'\rangle t''$ by the rule $(Perm_1)$.
Both reducts can be reduced to
$\langle xs/y\rangle [t/z]\langle \underline{z}s'/w\rangle [t'/z']\langle \underline{z'}s''/w'\rangle t''$ by some $\mathtt{x}$-reduction steps.

(b) $[[[r/x]\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t'/z']\langle \underline{z'}s''/w'\rangle t''$ is reduced to
$[[r/x][\langle \underline{x}s/y\rangle t/z]\langle \underline{z}s'/w\rangle t'/z']\langle \underline{z'}s''/w'\rangle t''$ by the rule $(Perm_1)$, or to
$[[r/x]\langle \underline{x}s/y\rangle t/z][\langle \underline{z}s'/w\rangle t'/z']\langle \underline{z'}s''/w'\rangle t''$ by the same rule in another way.
Both reducts can be reduced to
$[r/x]\langle \underline{x}s/y\rangle [t/z]\langle \underline{z}s'/w\rangle [t'/z']\langle \underline{z'}s''/w'\rangle t''$ by some $\mathtt{x}$-reduction steps.

Then the local confluence of $\mathtt{x}$-reduction follows by induction on the structure of terms. □

Lemma 3. *Let* u, t, s *be Beta-terms. Then*

1. $[u/x]t \overset{*}{\to}_{\mathtt{x}} \{u/x\}t$,
2. $[u/x]\langle \underline{x}s/y\rangle t \overset{*}{\to}_{\mathtt{x}} \langle \{\!\!\{u\}\!\!\} s/y\rangle t$. *Moreover,* $\langle \{\!\!\{u\}\!\!\} s/y\rangle t$ *is a Beta-term, hence* $\mathtt{x}([u/x]\langle \underline{x}s/y\rangle t) \equiv \langle \{\!\!\{u\}\!\!\} s/y\rangle t$.

Proof. 1. By induction on the structure of t.

(a) $t \equiv y$ $(y \not\equiv x)$. Then $[u/x]y \to_{\mathtt{x}} y \equiv \{u/x\}y$.
(b) $t \equiv x$. Then $[u/x]x \to_{\mathtt{x}} u \equiv \{u/x\}x$.
(c) $t \equiv \lambda y.t'$. Then $[u/x](\lambda y.t') \to_{\mathtt{x}} \lambda y.[u/x]t' \overset{\mathrm{IH}*}{\to}_{\mathtt{x}} \lambda y.\{u/x\}t' \equiv \{u/x\}(\lambda y.t')$.
(d) $t \equiv \langle zs/y\rangle t'$ $(z \not\equiv x)$. Then

$$\begin{aligned} [u/x]\langle zs/y\rangle t' &\to_{\mathtt{x}} \langle z([u/x]s)/y\rangle [u/x]t' \\ &\overset{*}{\to}_{\mathtt{x}} \langle z(\{u/x\}s)/y\rangle \{u/x\}t' && \text{(by IH)} \\ &\equiv \{u/x\}\langle zs/y\rangle t' \end{aligned}$$

(e) $t \equiv \langle xs/y\rangle t'$ $(x \in FV(s) \cup FV(t'))$. Then

$$\begin{aligned} [u/x]\langle xs/y\rangle t' &\to_{\mathtt{x}} [u/x]\langle \underline{x}([u/x]s)/y\rangle [u/x]t' \\ &\overset{*}{\to}_{\mathtt{x}} [u/x]\langle \underline{x}(\{u/x\}s)/y\rangle \{u/x\}t' && \text{(by IH)} \\ &\equiv \{u/x\}\langle xs/y\rangle t' \end{aligned}$$

(f) $t \equiv \langle \underline{x}s/y\rangle t'$. Then

$$\begin{aligned} [u/x]\langle \underline{x}s/y\rangle t' &\equiv [u/x]\langle \underline{x}(\{u/x\}s)/y\rangle \{u/x\}t' && \text{(by Lemma 1)} \\ &\equiv \{u/x\}\langle \underline{x}s/y\rangle t' \end{aligned}$$

(g) $t \equiv [\lambda z.r/w]\langle \underline{w}s/y\rangle t'$. Then

$$\begin{aligned}
[u/x][\lambda z.r/w]\langle \underline{w}s/y\rangle t' &\to_{\mathtt{x}} [[u/x](\lambda z.r)/w][u/x]\langle \underline{w}s/y\rangle t' \\
&\to_{\mathtt{x}} [\lambda z.[u/x]r/w][u/x]\langle \underline{w}s/y\rangle t' \\
&\to_{\mathtt{x}} [\lambda z.[u/x]r/w]\langle \underline{w}([u/x]s)/y\rangle [u/x]t' \\
&\xrightarrow{*}_{\mathtt{x}} [\lambda z.\{u/x\}r/w]\langle \underline{w}(\{u/x\}s)/y\rangle \{u/x\}t' \quad \text{(by IH)} \\
&\equiv [\{u/x\}(\lambda z.r)/w]\{u/x\}\langle \underline{w}s/y\rangle t' \\
&\equiv \{u/x\}[\lambda z.r/w]\langle \underline{w}s/y\rangle t'
\end{aligned}$$

2. By induction on the structure of u.

(a) $u \equiv z$. Then $[z/x]\langle \underline{x}s/y\rangle t \to_{\mathtt{x}} \langle zs/y\rangle t \equiv \langle \boldsymbol{\{} z\boldsymbol{\}} s/y\rangle t$.

(b) $u \equiv \lambda z.r$. Then $[\lambda z.r/x]\langle \underline{x}s/y\rangle t \equiv \langle \boldsymbol{\{}\lambda z.r\boldsymbol{\}} s/y\rangle t$.

(c) $u \equiv \langle zs'/w\rangle r$. Then

$$\begin{aligned}
[\langle zs'/w\rangle r/x]\langle \underline{x}s/y\rangle t &\to_{\mathtt{x}} \langle zs'/w\rangle [r/x]\langle \underline{x}s/y\rangle t \\
&\xrightarrow{*}_{\mathtt{x}} \langle zs'/w\rangle \langle \boldsymbol{\{} r\boldsymbol{\}} s/y\rangle t \quad \text{(by IH)} \\
&\equiv \langle \boldsymbol{\{}\langle zs'/w\rangle r\boldsymbol{\}} s/y\rangle t
\end{aligned}$$

(d) $u \equiv [\lambda z.r/w]\langle \underline{w}s'/y'\rangle t'$. Then

$$\begin{aligned}
[[\lambda z.r/w]\langle \underline{w}s'/y'\rangle t'/x]\langle \underline{x}s/y\rangle t &\to_{\mathtt{x}} [\lambda z.r/w][\langle \underline{w}s'/y'\rangle t'/x]\langle \underline{x}s/y\rangle t \\
&\to_{\mathtt{x}} [\lambda z.r/w]\langle \underline{w}s'/y'\rangle [t'/x]\langle \underline{x}s/y\rangle t \\
&\xrightarrow{*}_{\mathtt{x}} [\lambda z.r/w]\langle \underline{w}s'/y'\rangle \langle \boldsymbol{\{} t'\boldsymbol{\}} s/y\rangle t \quad \text{(by IH)} \\
&\equiv [\lambda z.r/w]\langle \boldsymbol{\{}\langle \underline{w}s'/y'\rangle t'\boldsymbol{\}} s/y\rangle t \\
&\equiv \langle \boldsymbol{\{}[\lambda z.r/w]\langle \underline{w}s'/y'\rangle t'\boldsymbol{\}} s/y\rangle t
\end{aligned}$$

□

Lemma 4. *Let u, s, t be Beta-terms. Then*

2. $\{u/x\}t \xrightarrow{*}_{\mathrm{gcut}} \mathtt{x}(\{u/x\}t)$.

Proof. 2. By induction on the structure of t.

(a) $t \equiv y$ ($y \not\equiv x$). Then $\{u/x\}y \equiv y \equiv \mathtt{x}(y) \equiv \mathtt{x}(\{u/x\}y)$.

(b) $t \equiv x$. Then $\{u/x\}x \equiv u \equiv \mathtt{x}(u) \equiv \mathtt{x}(\{u/x\}x)$.

(c) $t \equiv \lambda y.t'$. Then $\{u/x\}(\lambda y.t') \equiv \lambda y.\{u/x\}t' \xrightarrow{\mathrm{IH}*}_{\mathrm{gcut}} \lambda y.\mathtt{x}(\{u/x\}t') \equiv \mathtt{x}(\lambda y.\{u/x\}t') \equiv \mathtt{x}(\{u/x\}(\lambda y.t'))$.

(d) $t \equiv \langle zs/y\rangle t'$ ($z \not\equiv x$). Then

$$\begin{aligned}
\{u/x\}\langle zs/y\rangle t' &\equiv \langle z(\{u/x\}s)/y\rangle \{u/x\}t' \\
&\xrightarrow{*}_{\mathrm{gcut}} \langle z\mathtt{x}(\{u/x\}s)/y\rangle \mathtt{x}(\{u/x\}t') \quad \text{(by IH)} \\
&\equiv \mathtt{x}(\langle z\{u/x\}s/y\rangle \{u/x\}t') \\
&\equiv \mathtt{x}(\{u/x\}\langle zs/y\rangle t')
\end{aligned}$$

(e) $t \equiv \langle xs/y \rangle t'$. Then

$$\begin{aligned}
\{u/x\}\langle xs/y \rangle t' &\equiv [u/x]\langle \underline{x}(\{u/x\}s)/y \rangle \{u/x\}t' \\
&\xrightarrow{*}_{\text{gcut}} [u/x]\langle \underline{x}\mathtt{x}(\{u/x\}s)/y \rangle \mathtt{x}(\{u/x\}t') \qquad \text{(by IH)} \\
&\xrightarrow{*}_{\text{gcut}} \mathtt{x}([u/x]\langle \underline{x}\mathtt{x}(\{u/x\}s)/y \rangle \mathtt{x}(\{u/x\}t')) \\
&\qquad \text{(by Lemma 4 (1))} \\
&\equiv \mathtt{x}([u/x]\langle \underline{x}(\{u/x\}s)/y \rangle \{u/x\}t') \\
&\equiv \mathtt{x}(\{u/x\}\langle xs/y \rangle t')
\end{aligned}$$

(f) $t \equiv [\lambda z.r/w]\langle \underline{w}s/y \rangle t'$. Then

$$\begin{aligned}
&\{u/x\}[\lambda z.r/w]\langle \underline{w}s/y \rangle t' \\
&\equiv [\{u/x\}(\lambda z.r)/w]\{u/x\}\langle \underline{w}s/y \rangle t' \\
&\equiv [\lambda z.\{u/x\}r/w]\langle \underline{w}(\{u/x\}s)/y \rangle \{u/x\}t' \\
&\xrightarrow{*}_{\text{gcut}} [\lambda z.\mathtt{x}(\{u/x\}r)/w]\langle \underline{w}\mathtt{x}(\{u/x\}s)/y \rangle \mathtt{x}(\{u/x\}t') \qquad \text{(by IH)} \\
&\equiv \mathtt{x}([\lambda z.\{u/x\}r/w]\langle \underline{w}\{u/x\}s/y \rangle \{u/x\}t') \\
&\equiv \mathtt{x}(\{u/x\}[\lambda z.r/w]\langle \underline{w}s/y \rangle t')
\end{aligned}$$

□

Lemma 5. $t \xrightarrow{*}_{\text{gcut}} \mathtt{x}(t)$.

Proof. By induction on the structure of t. We treat the case $t \equiv [u/x]r$. Then by the induction hypothesis, $u \xrightarrow{*}_{\text{gcut}} \mathtt{x}(u)$ and $r \xrightarrow{*}_{\text{gcut}} \mathtt{x}(r)$. Hence $[u/x]r \xrightarrow{*}_{\text{gcut}} [\mathtt{x}(u)/x]\mathtt{x}(r)$. Now we consider three cases.

(a) If $\mathtt{x}(u) \equiv \lambda z.r'$ and $\mathtt{x}(r) \equiv \langle \underline{x}s/y \rangle t'$, then $[\mathtt{x}(u)/x]\mathtt{x}(r)$ is in x-normal form. Since $[u/x]r \xrightarrow{*}_{\mathtt{x}} [\mathtt{x}(u)/x]\mathtt{x}(r)$, we have $[\mathtt{x}(u)/x]\mathtt{x}(r) \equiv \mathtt{x}([u/x]r)$.

(b) If $\mathtt{x}(u)$ is not of the form $\lambda z.r'$ and $\mathtt{x}(r) \equiv \langle \underline{x}s/y \rangle t'$, then

$$\begin{aligned}
[\mathtt{x}(u)/x]\mathtt{x}(r) &\equiv [\mathtt{x}(u)/x]\langle \underline{x}s/y \rangle t' \\
&\to_{\mathit{left}} \langle \{\mathtt{x}(u)\}s/y \rangle t' \\
&\equiv \mathtt{x}([\mathtt{x}(u)/x]\langle \underline{x}s/y \rangle t') \qquad \text{(by Lemma 3 (2))} \\
&\equiv \mathtt{x}([\mathtt{x}(u)/x]\mathtt{x}(r)) \\
&\equiv \mathtt{x}([u/x]r)
\end{aligned}$$

(c) If $\mathtt{x}(r)$ is not of the form $\langle \underline{x}s/y \rangle t'$, then

$$\begin{aligned}
[\mathtt{x}(u)/x]\mathtt{x}(r) &\to_{\mathit{right}} \{\mathtt{x}(u)/x\}\mathtt{x}(r) \\
&\xrightarrow{*}_{\text{gcut}} \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(r)) \qquad \text{(by Lemma 4 (2))} \\
&\equiv \mathtt{x}([\mathtt{x}(u)/x]\mathtt{x}(r)) \qquad \text{(by Lemma 3 (1))} \\
&\equiv \mathtt{x}([u/x]r)
\end{aligned}$$

□

Lemma 6. $\langle \pmb{\{}\langle \pmb{\{}u\pmb{\}} s/y\rangle t\pmb{\}} s'/y'\rangle t' \equiv \langle \pmb{\{}u\pmb{\}} s/y\rangle \langle \pmb{\{}t\pmb{\}} s'/y'\rangle t'$.

Proof. By induction on the structure of u.

(a) $u \equiv z$. Then

$$\begin{aligned}
\langle \pmb{\{}\langle \pmb{\{}z\pmb{\}} s/y\rangle t\pmb{\}} s'/y'\rangle t' &\equiv \langle \pmb{\{}\langle zs/y\rangle t\pmb{\}} s'/y'\rangle t' \\
&\equiv \langle zs/y\rangle \langle \pmb{\{}t\pmb{\}} s'/y'\rangle t' \\
&\equiv \langle \pmb{\{}z\pmb{\}} s/y\rangle \langle \pmb{\{}t\pmb{\}} s'/y'\rangle t'
\end{aligned}$$

(b) $u \equiv \lambda z.r$. Then

$$\begin{aligned}
\langle \pmb{\{}\langle \pmb{\{}\lambda z.r\pmb{\}} s/y\rangle t\pmb{\}} s'/y'\rangle t' &\equiv \langle \pmb{\{}[\lambda z.r/x]\langle \underline{x} s/y\rangle t\pmb{\}} s'/y'\rangle t' \\
&\equiv [\lambda z.r/x]\langle \pmb{\{}\langle \underline{x} s/y\rangle t\pmb{\}} s'/y'\rangle t' \\
&\equiv [\lambda z.r/x]\langle \underline{x} s/y\rangle \langle \pmb{\{}t\pmb{\}} s'/y'\rangle t' \\
&\equiv \langle \pmb{\{}\lambda z.r\pmb{\}} s/y\rangle \langle \pmb{\{}t\pmb{\}} s'/y'\rangle t'
\end{aligned}$$

(c) $u \equiv \langle zs_0/w\rangle r$. Then

$$\begin{aligned}
\langle \pmb{\{}\langle \pmb{\{}\langle zs_0/w\rangle r\pmb{\}} s/y\rangle t\pmb{\}} s'/y'\rangle t' &\equiv \langle \pmb{\{}\langle zs_0/w\rangle \langle \pmb{\{}r\pmb{\}} s/y\rangle t\pmb{\}} s'/y'\rangle t' \\
&\equiv \langle zs_0/w\rangle \langle \pmb{\{}\langle \pmb{\{}r\pmb{\}} s/y\rangle t\pmb{\}} s'/y'\rangle t' \\
&\equiv \langle zs_0/w\rangle \langle \pmb{\{}r\pmb{\}} s/y\rangle \langle \pmb{\{}t\pmb{\}} s'/y'\rangle t' \qquad \text{(by IH)} \\
&\equiv \langle \pmb{\{}\langle zs_0/w\rangle r\pmb{\}} s/y\rangle \langle \pmb{\{}t\pmb{\}} s'/y'\rangle t'
\end{aligned}$$

(d) $u \equiv [s_0/w]r$. Then

$$\begin{aligned}
\langle \pmb{\{}\langle \pmb{\{}[s_0/w]r\pmb{\}} s/y\rangle t\pmb{\}} s'/y'\rangle t' &\equiv \langle \pmb{\{}[s_0/w]\langle \pmb{\{}r\pmb{\}} s/y\rangle t\pmb{\}} s'/y'\rangle t' \\
&\equiv [s_0/w]\langle \pmb{\{}\langle \pmb{\{}r\pmb{\}} s/y\rangle t\pmb{\}} s'/y'\rangle t' \\
&\equiv [s_0/w]\langle \pmb{\{}r\pmb{\}} s/y\rangle \langle \pmb{\{}t\pmb{\}} s'/y'\rangle t' \qquad \text{(by IH)} \\
&\equiv \langle \pmb{\{}[s_0/w]r\pmb{\}} s/y\rangle \langle \pmb{\{}t\pmb{\}} s'/y'\rangle t'
\end{aligned}$$

□

In the proofs below, we often use the notation ${}^{\mathtt{x}}\{u/x\}t$ for $\mathtt{x}(\{u/x\}t)$. The following facts are useful.

Proposition 5.

1. ${}^{\mathtt{x}}\{u/x\}(\lambda z.t) \equiv \lambda z.{}^{\mathtt{x}}\{u/x\}t$,
2. ${}^{\mathtt{x}}\{u/x\}\langle zs/y\rangle t \equiv \langle z\,{}^{\mathtt{x}}\{u/x\}s/y\rangle {}^{\mathtt{x}}\{u/x\}t \ \ (z \not\equiv x)$,
3. ${}^{\mathtt{x}}\{u/x\}[\lambda z.r/w]\langle \underline{w}s/y\rangle t \equiv [\lambda z.{}^{\mathtt{x}}\{u/x\}r/w]\langle \underline{w}\,{}^{\mathtt{x}}\{u/x\}s/y\rangle {}^{\mathtt{x}}\{u/x\}t$.

Lemma 7. *Let u, t, s, t' be Beta-terms. Then*
$\mathtt{x}(\{u/x\}\langle \pmb{\{}t\pmb{\}} s/y\rangle t') \equiv \langle \pmb{\{}\mathtt{x}(\{u/x\}t)\pmb{\}} \mathtt{x}(\{u/x\}s)/y\rangle \mathtt{x}(\{u/x\}t')$.
In particular, if $x \notin FV(s) \cup FV(t')$ then
$\mathtt{x}(\{u/x\}\langle \pmb{\{}t\pmb{\}} s/y\rangle t') \equiv \langle \pmb{\{}\mathtt{x}(\{u/x\}t)\pmb{\}} s/y\rangle t'$.

Proof. By induction on the structure of t. Here we prove the latter claim. The former is proved in a similar way.

(a) $t \equiv z$ $(z \not\equiv x)$. Then

$$\begin{aligned}
\mathtt{x}(\{u/x\}\langle \{\!\{z\}\!\} s/y\rangle t') &\equiv \mathtt{x}(\{u/x\}\langle zs/y\rangle t') \\
&\equiv \mathtt{x}(\langle zs/y\rangle t') && \text{(by Lemma 1)} \\
&\equiv \langle zs/y\rangle t' \\
&\equiv \langle \{\!\{z\}\!\} s/y\rangle t' \\
&\equiv \langle \{\!\{\mathtt{x}(z)\}\!\} s/y\rangle t' \\
&\equiv \langle \{\!\{\mathtt{x}(\{u/x\}z)\}\!\} s/y\rangle t'
\end{aligned}$$

(b) $t \equiv x$. Then

$$\begin{aligned}
\mathtt{x}(\{u/x\}\langle \{\!\{x\}\!\} s/y\rangle t') &\equiv \mathtt{x}(\{u/x\}\langle xs/y\rangle t') \\
&\equiv \mathtt{x}([u/x]\langle \underline{x}(\{u/x\}s)/y\rangle \{u/x\}t') \\
&\equiv \mathtt{x}([u/x]\langle \underline{x}s/y\rangle t') && \text{(by Lemma 1)} \\
&\equiv \langle \{\!\{u\}\!\} s/y\rangle t' && \text{(by Lemma 3 (2))} \\
&\equiv \langle \{\!\{\mathtt{x}(u)\}\!\} s/y\rangle t' \\
&\equiv \langle \{\!\{\mathtt{x}(\{u/x\}x)\}\!\} s/y\rangle t'
\end{aligned}$$

(c) $t \equiv \lambda z.r$. Then

$$\begin{aligned}
\mathtt{x}(\{u/x\}\langle \{\!\{\lambda z.r\}\!\} s/y\rangle t') &\equiv \mathtt{x}(\{u/x\}[\lambda z.r/w]\langle \underline{w}s/y\rangle t') \\
&\equiv \mathtt{x}([\{u/x\}(\lambda z.r)/w]\{u/x\}\langle \underline{w}s/y\rangle t') \\
&\equiv \mathtt{x}([\{u/x\}(\lambda z.r)/w]\langle \underline{w}s/y\rangle t') && \text{(by Lemma 1)} \\
&\equiv \mathtt{x}([\mathtt{x}(\{u/x\}(\lambda z.r))/w]\langle \underline{w}s/y\rangle t') \\
&\equiv \langle \{\!\{\mathtt{x}(\{u/x\}(\lambda z.r))\}\!\} s/y\rangle t' && \text{(by Lemma 3 (2))}
\end{aligned}$$

(d) $t \equiv \langle zs'/w\rangle r$ $(z \not\equiv x)$. Then

$$\begin{aligned}
{}^{\mathtt{x}}\{u/x\}\langle \{\!\{\langle zs'/w\rangle r\}\!\} s/y\rangle t' &\equiv {}^{\mathtt{x}}\{u/x\}\langle zs'/w\rangle \langle \{\!\{r\}\!\} s/y\rangle t' \\
&\equiv \langle z\,{}^{\mathtt{x}}\{u/x\}s'/w\rangle {}^{\mathtt{x}}\{u/x\}\langle \{\!\{r\}\!\} s/y\rangle t' \\
&\equiv \langle z\,{}^{\mathtt{x}}\{u/x\}s'/w\rangle \langle \{\!\{{}^{\mathtt{x}}\{u/x\}r\}\!\} s/y\rangle t' && \text{(by IH)} \\
&\equiv \langle \{\!\{\langle z\,{}^{\mathtt{x}}\{u/x\}s'/w\rangle {}^{\mathtt{x}}\{u/x\}r\}\!\} s/y\rangle t' \\
&\equiv \langle \{\!\{{}^{\mathtt{x}}\{u/x\}\langle zs'/w\rangle r\}\!\} s/y\rangle t'
\end{aligned}$$

(e) $t \equiv \langle xs'/w\rangle r$. Then

$$
\begin{aligned}
& \mathtt{x}(\{u/x\}\langle \{\!\{\langle xs'/w\rangle r\}\!\} s/y\rangle t') \\
\equiv\ & \mathtt{x}(\{u/x\}\langle xs'/w\rangle\langle \{\!\{r\}\!\} s/y\rangle t') \\
\equiv\ & \mathtt{x}([u/x]\langle \underline{x}\{u/x\}s'/w\rangle\{u/x\}\langle \{\!\{r\}\!\} s/y\rangle t') \\
\equiv\ & \mathtt{x}([u/x]\langle \underline{x}{}^{\mathtt{x}}\{u/x\}s'/w\rangle{}^{\mathtt{x}}\{u/x\}\langle \{\!\{r\}\!\} s/y\rangle t') \\
\equiv\ & \langle \{\!\{u\}\!\}{}^{\mathtt{x}}\{u/x\}s'/w\rangle{}^{\mathtt{x}}\{u/x\}\langle \{\!\{r\}\!\} s/y\rangle t' && \text{(by Lemma 3 (2))} \\
\equiv\ & \langle \{\!\{u\}\!\}{}^{\mathtt{x}}\{u/x\}s'/w\rangle\langle \{\!\{{}^{\mathtt{x}}\{u/x\}r\}\!\} s/y\rangle t' && \text{(by IH)} \\
\equiv\ & \langle \{\!\{\langle \{\!\{u\}\!\}{}^{\mathtt{x}}\{u/x\}s'/w\rangle{}^{\mathtt{x}}\{u/x\}r\}\!\} s/y\rangle t' && \text{(by Lemma 6)} \\
\equiv\ & \langle \{\!\{\mathtt{x}([u/x]\langle \underline{x}{}^{\mathtt{x}}\{u/x\}s'/w\rangle{}^{\mathtt{x}}\{u/x\}r)\}\!\} s/y\rangle t' && \text{(by Lemma 3 (2))} \\
\equiv\ & \langle \{\!\{\mathtt{x}([u/x]\langle \underline{x}\{u/x\}s'/w\rangle\{u/x\}r)\}\!\} s/y\rangle t' \\
\equiv\ & \langle \{\!\{\mathtt{x}(\{u/x\}\langle xs'/w\rangle r)\}\!\} s/y\rangle t'
\end{aligned}
$$

(f) $t \equiv [\lambda z.r/w]\langle \underline{w}s'/y'\rangle t''$. Then

$$
\begin{aligned}
& {}^{\mathtt{x}}\{u/x\}\langle \{\!\{[\lambda z.r/w]\langle \underline{w}s'/y'\rangle t''\}\!\} s/y\rangle t' \\
\equiv\ & {}^{\mathtt{x}}\{u/x\}[\lambda z.r/w]\langle \underline{w}s'/y'\rangle\langle \{\!\{t''\}\!\} s/y\rangle t' \\
\equiv\ & [\lambda z.{}^{\mathtt{x}}\{u/x\}r/w]\langle \underline{w}{}^{\mathtt{x}}\{u/x\}s'/y'\rangle{}^{\mathtt{x}}\{u/x\}\langle \{\!\{t''\}\!\} s/y\rangle t' \\
\equiv\ & [\lambda z.{}^{\mathtt{x}}\{u/x\}r/w]\langle \underline{w}{}^{\mathtt{x}}\{u/x\}s'/y'\rangle\langle \{\!\{{}^{\mathtt{x}}\{u/x\}t''\}\!\} s/y\rangle t' && \text{(by IH)} \\
\equiv\ & \langle \{\!\{[\lambda z.{}^{\mathtt{x}}\{u/x\}r/w]\langle \underline{w}{}^{\mathtt{x}}\{u/x\}s'/y'\rangle{}^{\mathtt{x}}\{u/x\}t''\}\!\} s/y\rangle t' \\
\equiv\ & \langle \{\!\{{}^{\mathtt{x}}\{u/x\}[\lambda z.r/w]\langle \underline{w}s'/y'\rangle t''\}\!\} s/y\rangle t'
\end{aligned}
$$

□

Lemma 8. *Let u, s, t be Beta-terms with $y \notin FV(u)$. Then* $\mathtt{x}(\{u/x\}\mathtt{x}(\{s/y\}t)) \equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}\mathtt{x}(\{u/x\}t))$.

Proof. By induction on the structure of t.

(a) $t \equiv z$ ($z \not\equiv x, y$). Then

$$
\begin{aligned}
\mathtt{x}(\{u/x\}\mathtt{x}(\{s/y\}z)) &\equiv \mathtt{x}(\{u/x\}\mathtt{x}(z)) \\
&\equiv \mathtt{x}(\{u/x\}z) \\
&\equiv \mathtt{x}(z) \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}z) \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}\mathtt{x}(z)) \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}\mathtt{x}(\{u/x\}z))
\end{aligned}
$$

(b) $t \equiv y$. Then

$$
\begin{aligned}
\mathtt{x}(\{u/x\}\mathtt{x}(\{s/y\}y)) &\equiv \mathtt{x}(\{u/x\}\mathtt{x}(s)) \\
&\equiv \mathtt{x}(\{u/x\}s) \\
&\equiv \mathtt{x}(\mathtt{x}(\{u/x\}s)) \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}y) \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}\mathtt{x}(y)) \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}\mathtt{x}(\{u/x\}y))
\end{aligned}
$$

(c) $t \equiv x$. Then

$$\begin{aligned}
\mathtt{x}(\{u/x\}\mathtt{x}(\{s/y\}x)) &\equiv \mathtt{x}(\{u/x\}\mathtt{x}(x)) \\
&\equiv \mathtt{x}(\{u/x\}x) \\
&\equiv \mathtt{x}(u) \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}u) && \text{(by Lemma 1)} \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}\mathtt{x}(u)) \\
&\equiv \mathtt{x}(\{\mathtt{x}(\{u/x\}s)/y\}\mathtt{x}(\{u/x\}x))
\end{aligned}$$

(d) $t \equiv \lambda z.t'$. Then

$$\begin{aligned}
{}^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}(\lambda z.t') &\equiv {}^{\mathtt{x}}\{u/x\}(\lambda z.{}^{\mathtt{x}}\{s/y\}t') \\
&\equiv \lambda z.{}^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}t' \\
&\equiv \lambda z.{}^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}t' && \text{(by IH)} \\
&\equiv {}^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}(\lambda z.{}^{\mathtt{x}}\{u/x\}t') \\
&\equiv {}^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}(\lambda z.t')
\end{aligned}$$

(e) $t \equiv \langle zs'/y'\rangle t'$ $(z \not\equiv x, y)$. Then

$$\begin{aligned}
&{}^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}\langle zs'/y'\rangle t' \\
&\equiv {}^{\mathtt{x}}\{u/x\}\langle z^{\mathtt{x}}\{s/y\}s'/y'\rangle^{\mathtt{x}}\{s/y\}t' \\
&\equiv \langle z^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}s'/y'\rangle^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}t' \\
&\equiv \langle z^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}s'/y'\rangle^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}t' && \text{(by IH)} \\
&\equiv {}^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}\langle z^{\mathtt{x}}\{u/x\}s'/y'\rangle^{\mathtt{x}}\{u/x\}t' \\
&\equiv {}^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}\langle zs'/y'\rangle t'
\end{aligned}$$

(f) $t \equiv \langle ys'/y'\rangle t'$. Then

$$\begin{aligned}
&{}^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}\langle ys'/y'\rangle t' \\
&\equiv {}^{\mathtt{x}}\{u/x\}\langle \boldsymbol{\{} s \boldsymbol{\}}^{\mathtt{x}}\{s/y\}s'/y'\rangle^{\mathtt{x}}\{s/y\}t' && \text{(by Lemma 3 (2))} \\
&\equiv \langle \boldsymbol{\{}{}^{\mathtt{x}}\{u/x\}s\boldsymbol{\}}^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}s'/y'\rangle^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}t' && \text{(by Lemma 7)} \\
&\equiv \langle \boldsymbol{\{}{}^{\mathtt{x}}\{u/x\}s\boldsymbol{\}}^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}s'/y'\rangle^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}t' && \text{(by IH)} \\
&\equiv {}^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}\langle y^{\mathtt{x}}\{u/x\}s'/y'\rangle^{\mathtt{x}}\{u/x\}t' && \text{(by Lemma 3 (2))} \\
&\equiv {}^{\mathtt{x}}\{{}^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}\langle ys'/y'\rangle t'
\end{aligned}$$

(g) $t \equiv \langle xs'/y'\rangle t'$. Then

$$\begin{aligned}
&{}^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}\langle xs'/y'\rangle t' \\
\equiv\ & {}^{\mathtt{x}}\{u/x\}\langle x^{\mathtt{x}}\{s/y\}s'/y'\rangle^{\mathtt{x}}\{s/y\}t' \\
\equiv\ & \langle \{\!\!\{u\}\!\!\}^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}s'/y'\rangle^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}t' && \text{(by Lemma 3 (2))} \\
\equiv\ & \langle \{\!\!\{u\}\!\!\}^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}s'/y'\rangle^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}t' && \text{(by IH)} \\
\equiv\ & \langle \{\!\!\{^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}u\}\!\!\}^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}s'/y'\rangle^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}t' \\
& && \text{(by Lemma 1)} \\
\equiv\ & {}^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}\langle \{\!\!\{u\}\!\!\}^{\mathtt{x}}\{u/x\}s'/y'\rangle^{\mathtt{x}}\{u/x\}t' && \text{(by Lemma 7)} \\
\equiv\ & {}^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}\langle xs'/y'\rangle t' && \text{(by Lemma 3 (2))}
\end{aligned}$$

(h) $t \equiv [\lambda z.r/w]\langle \underline{w}s'/y'\rangle t''$. Then

$$\begin{aligned}
&{}^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}[\lambda z.r/w]\langle \underline{w}s'/y'\rangle t' \\
\equiv\ & {}^{\mathtt{x}}\{u/x\}[\lambda z.^{\mathtt{x}}\{s/y\}r/w]\langle \underline{w}^{\mathtt{x}}\{s/y\}s'/y'\rangle^{\mathtt{x}}\{s/y\}t' \\
\equiv\ & [\lambda z.^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}r/w]\langle \underline{w}^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}s'/y'\rangle^{\mathtt{x}}\{u/x\}^{\mathtt{x}}\{s/y\}t' \\
\equiv\ & [\lambda z.^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}r/w]\langle \underline{w}^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}s'/y'\rangle^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}t' \\
& && \text{(by IH)} \\
\equiv\ & {}^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}[\lambda z.^{\mathtt{x}}\{u/x\}r/w]\langle \underline{w}^{\mathtt{x}}\{u/x\}s'/y'\rangle^{\mathtt{x}}\{u/x\}t' \\
\equiv\ & {}^{\mathtt{x}}\{^{\mathtt{x}}\{u/x\}s/y\}^{\mathtt{x}}\{u/x\}[\lambda z.r/w]\langle \underline{w}s'/y'\rangle t'
\end{aligned}$$

□

C Proofs in Section 4

In this appendix we give proofs of some lemmas in Section 4.

Lemma 10.

2. *If* $t \Rightarrow t'$ *then* $t \xrightarrow{*}_{\beta} t'$.
3. *If* $u \Rightarrow u'$, $s \Rightarrow s'$ *and* $t \Rightarrow t'$ *then* $\langle \{\!\!\{u\}\!\!\}s/y\rangle t \Rightarrow \langle \{\!\!\{u'\}\!\!\}s'/y\rangle t'$.
4. *If* $u \Rightarrow u'$ *and* $t \Rightarrow t'$ *then* $\mathtt{x}(\{u/x\}t) \Rightarrow \mathtt{x}(\{u'/x\}t')$.

Proof. 2. By induction on the definition of $t \Rightarrow t'$. We treat the case (pr_5). Then by the induction hypothesis, $r \xrightarrow{*}_{\beta} r'$, $s \xrightarrow{*}_{\beta} s'$ and $t \xrightarrow{*}_{\beta} t'$. Hence $[\lambda z.r/x]\langle \underline{x}s/y\rangle t \xrightarrow{*}_{\beta} [\lambda z.r'/x]\langle \underline{x}s'/y\rangle t' \to_{\beta} \mathtt{x}(\{\mathtt{x}(\{s'/z\}r')/y\}t')$.

3. By induction on the definition of $u \Rightarrow u'$. We treat the case (pr_5). Let $u \equiv [\lambda z.r_0/x]\langle \underline{x}s_0/y_0\rangle t_0 \Rightarrow \mathtt{x}(\{\mathtt{x}(\{s_0'/z\}r_0')/y_0\}t_0') \equiv u'$ $(y_0 \notin FV(s') \cup FV(t'))$. Then by the induction hypothesis, $\langle \{\!\!\{t_0\}\!\!\}s/y\rangle t \Rightarrow \langle \{\!\!\{t_0'\}\!\!\}s'/y\rangle t'$, so by (pr_5), $[\lambda z.r_0/x]\langle \underline{x}s_0/y_0\rangle\langle \{\!\!\{t_0\}\!\!\}s/y\rangle t \Rightarrow \mathtt{x}(\{\mathtt{x}(\{s_0'/z\}r_0')/y_0\}\langle \{\!\!\{t_0'\}\!\!\}s'/y\rangle t')$, where
LHS $\equiv \langle \{\!\!\{[\lambda z.r_0/x]\langle \underline{x}s_0/y_0\rangle t_0\}\!\!\}s/y\rangle t$ and
RHS $\equiv \langle \{\!\!\{\mathtt{x}(\{\mathtt{x}(\{s_0'/z\}r_0')/y_0\}t_0')\}\!\!\}s'/y\rangle t'$ by Lemma 7.
Hence $\langle \{\!\!\{[\lambda z.r_0/x]\langle \underline{x}s_0/y_0\rangle t_0\}\!\!\}s/y\rangle t \Rightarrow \langle \{\!\!\{\mathtt{x}(\{\mathtt{x}(\{s_0'/z\}r_0')/y_0\}t_0')\}\!\!\}s'/y\rangle t'$.

4. By induction on the definition of $t \Rightarrow t'$. We treat some cases.

(pr_5) Let $t \equiv [\lambda z.r/w]\langle \underline{w}s/y\rangle t_0 \Rightarrow {}^{\mathtt{x}}\{{}^{\mathtt{x}}\{s'/z\}r'/y\}t'_0 \equiv t'$ ($y, z \notin FV(u')$). By the induction hypothesis, ${}^{\mathtt{x}}\{u/x\}r \Rightarrow {}^{\mathtt{x}}\{u'/x\}r'$, ${}^{\mathtt{x}}\{u/x\}s \Rightarrow {}^{\mathtt{x}}\{u'/x\}s'$ and ${}^{\mathtt{x}}\{u/x\}t_0 \Rightarrow {}^{\mathtt{x}}\{u'/x\}t'_0$, so by ($pr_5$), we have
$[\lambda z.{}^{\mathtt{x}}\{u/x\}r/w]\langle \underline{w}{}^{\mathtt{x}}\{u/x\}s/y\rangle {}^{\mathtt{x}}\{u/x\}t_0 \Rightarrow {}^{\mathtt{x}}\{{}^{\mathtt{x}}\{{}^{\mathtt{x}}\{u'/x\}s'/z\}{}^{\mathtt{x}}\{u'/x\}r'/y\}{}^{\mathtt{x}}\{u'/x\}t'_0$.
Here, LHS $\equiv {}^{\mathtt{x}}\{u/x\}[\lambda z.r/w]\langle \underline{w}s/y\rangle t_0$ and
RHS $\equiv {}^{\mathtt{x}}\{u'/x\}{}^{\mathtt{x}}\{{}^{\mathtt{x}}\{s'/z\}r'/y\}t'_0$ by applying Lemma 8 twice.
Hence ${}^{\mathtt{x}}\{u/x\}[\lambda z.r/w]\langle \underline{w}s/y\rangle t_0 \Rightarrow {}^{\mathtt{x}}\{u'/x\}{}^{\mathtt{x}}\{{}^{\mathtt{x}}\{s'/z\}r'/y\}t'_0$.

(pr_3) Let $t \equiv \langle xs/y\rangle t_0 \Rightarrow \langle xs'/y\rangle t'_0 \equiv t'$. Then by the induction hypothesis, ${}^{\mathtt{x}}\{u/x\}s \Rightarrow {}^{\mathtt{x}}\{u'/x\}s'$ and ${}^{\mathtt{x}}\{u/x\}t_0 \Rightarrow {}^{\mathtt{x}}\{u'/x\}t'_0$, so by Lemma 10 (3), we have $\langle \{\!\!\{u\}\!\!\}{}^{\mathtt{x}}\{u/x\}s/y\rangle {}^{\mathtt{x}}\{u/x\}t_0 \Rightarrow \langle \{\!\!\{u'\}\!\!\}{}^{\mathtt{x}}\{u'/x\}s'/y\rangle {}^{\mathtt{x}}\{u'/x\}t'_0$.
Here, LHS $\equiv {}^{\mathtt{x}}\{u/x\}\langle xs/y\rangle t_0$ by Lemma 3 (2), and
RHS $\equiv {}^{\mathtt{x}}\{u'/x\}\langle xs'/y\rangle t'_0$ by Lemma 3 (2).
Hence ${}^{\mathtt{x}}\{u/x\}\langle xs/y\rangle t_0 \Rightarrow {}^{\mathtt{x}}\{u'/x\}\langle xs'/y\rangle t'_0$. □

Lemma 11. *If $t \Rightarrow t'$ then $t' \Rightarrow t^\star$.*

Proof. By induction on the definition of $t \Rightarrow t'$, using Lemma 10 (4) in the case (pr_5). □

D Proofs in Section 5

In this appendix we give proofs of some lemmas in Section 5.

Lemma 14.

1. $\mathtt{x}(\langle \{\!\!\{u\}\!\!\}s/y\rangle t) \equiv \langle \{\!\!\{\mathtt{x}(u)\}\!\!\}\mathtt{x}(s)/y\rangle \mathtt{x}(t)$,
2. $\mathtt{x}(\{u/x\}t) \equiv \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(t))$.

Proof. 1. By induction on the structure of u.

(a) $u \equiv z$. Then

$$\begin{aligned}\mathtt{x}(\langle \{\!\!\{z\}\!\!\}s/y\rangle t) &\equiv \mathtt{x}(\langle zs/y\rangle t)\\ &\equiv \langle z\mathtt{x}(s)/y\rangle \mathtt{x}(t)\\ &\equiv \langle \{\!\!\{z\}\!\!\}\mathtt{x}(s)/y\rangle \mathtt{x}(t)\\ &\equiv \langle \{\!\!\{\mathtt{x}(z)\}\!\!\}\mathtt{x}(s)/y\rangle \mathtt{x}(t)\end{aligned}$$

(b) $u \equiv \lambda z.r$. Then

$$\begin{aligned}\mathtt{x}(\langle \{\!\!\{\lambda z.r\}\!\!\}s/y\rangle t) &\equiv \mathtt{x}([\lambda z.r/x]\langle \underline{x}s/y\rangle t)\\ &\equiv [\lambda z.\mathtt{x}(r)/x]\langle \underline{x}\mathtt{x}(s)/y\rangle \mathtt{x}(t)\\ &\equiv [\mathtt{x}(\lambda z.r)/x]\langle \underline{x}\mathtt{x}(s)/y\rangle \mathtt{x}(t)\\ &\equiv \langle \{\!\!\{\mathtt{x}(\lambda z.r)\}\!\!\}\mathtt{x}(s)/y\rangle \mathtt{x}(t) \qquad \text{(by Lemma 3 (2))}\end{aligned}$$

(c) $u \equiv \langle zs'/w\rangle r$. Then

$$\begin{aligned}
\mathtt{x}(\langle\{\!\{\langle zs'/w\rangle r\}\!\}s/y\rangle t) &\equiv \mathtt{x}(\langle zs'/w\rangle\langle\{\!\{r\}\!\}s/y\rangle t) \\
&\equiv \langle z\mathtt{x}(s')/w\rangle\mathtt{x}(\langle\{\!\{r\}\!\}s/y\rangle t) \\
&\equiv \langle z\mathtt{x}(s')/w\rangle\langle\{\!\{\mathtt{x}(r)\}\!\}\mathtt{x}(s)/y\rangle\mathtt{x}(t) && \text{(by IH)} \\
&\equiv \langle\{\!\{\langle z\mathtt{x}(s')/w\rangle\mathtt{x}(r)\}\!\}\mathtt{x}(s)/y\rangle\mathtt{x}(t) \\
&\equiv \langle\{\!\{\mathtt{x}(\langle zs'/w\rangle r)\}\!\}\mathtt{x}(s)/y\rangle\mathtt{x}(t)
\end{aligned}$$

(d) $u \equiv [s'/w]r$. We assume $w \notin FV(\mathtt{x}(s)) \cup FV(\mathtt{x}(t))$. Then

$$\begin{aligned}
&\mathtt{x}(\langle\{\!\{[s'/w]r\}\!\}s/y\rangle t) \\
\equiv\ &\mathtt{x}([s'/w]\langle\{\!\{r\}\!\}s/y\rangle t) \\
\equiv\ &\mathtt{x}([\mathtt{x}(s')/w]\mathtt{x}(\langle\{\!\{r\}\!\}s/y\rangle t)) \\
\equiv\ &\mathtt{x}(\{\mathtt{x}(s')/w\}\mathtt{x}(\langle\{\!\{r\}\!\}s/y\rangle t)) && \text{(by Lemma 3 (1))} \\
\equiv\ &\mathtt{x}(\{\mathtt{x}(s')/w\}\langle\{\!\{\mathtt{x}(r)\}\!\}\mathtt{x}(s)/y\rangle\mathtt{x}(t)) && \text{(by IH)} \\
\equiv\ &\langle\{\!\{\mathtt{x}(\{\mathtt{x}(s')/w\}\mathtt{x}(r))\}\!\}\mathtt{x}(s)/y\rangle\mathtt{x}(t) && \text{(by Lemma 7)} \\
\equiv\ &\langle\{\!\{\mathtt{x}([\mathtt{x}(s')/w]\mathtt{x}(r))\}\!\}\mathtt{x}(s)/y\rangle\mathtt{x}(t) && \text{(by Lemma 3 (1))} \\
\equiv\ &\langle\{\!\{\mathtt{x}([s'/w]r)\}\!\}\mathtt{x}(s)/y\rangle\mathtt{x}(t)
\end{aligned}$$

2. By induction on the structure of t.

(a) $t \equiv y$ $(y \not\equiv x)$. Then

$$\begin{aligned}
\mathtt{x}(\{u/x\}y) &\equiv \mathtt{x}(y) \\
&\equiv \mathtt{x}(\{\mathtt{x}(u)/x\}y) \\
&\equiv \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(y))
\end{aligned}$$

(b) $t \equiv x$. Then

$$\begin{aligned}
\mathtt{x}(\{u/x\}x) &\equiv \mathtt{x}(u) \\
&\equiv \mathtt{x}(\mathtt{x}(u)) \\
&\equiv \mathtt{x}(\{\mathtt{x}(u)/x\}x) \\
&\equiv \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(x))
\end{aligned}$$

(c) $t \equiv \lambda y.t'$. Then

$$\begin{aligned}
{}^{\mathtt{x}}\{u/x\}(\lambda y.t') &\equiv \lambda y.{}^{\mathtt{x}}\{u/x\}t' \\
&\equiv \lambda y.{}^{\mathtt{x}}\{\mathtt{x}(u)/x\}\mathtt{x}(t') && \text{(by IH)} \\
&\equiv {}^{\mathtt{x}}\{\mathtt{x}(u)/x\}(\lambda y.\mathtt{x}(t')) \\
&\equiv {}^{\mathtt{x}}\{\mathtt{x}(u)/x\}\mathtt{x}(\lambda y.t')
\end{aligned}$$

(d) $t \equiv \langle zs/y\rangle t'$ $(z \not\equiv x)$. Then

$$\begin{aligned}
{}^{\mathtt{x}}\{u/x\}\langle zs/y\rangle t' &\equiv \langle z\,{}^{\mathtt{x}}\{u/x\}s/y\rangle{}^{\mathtt{x}}\{u/x\}t' \\
&\equiv \langle z\,{}^{\mathtt{x}}\{\mathtt{x}(u)/x\}\mathtt{x}(s)/y\rangle{}^{\mathtt{x}}\{\mathtt{x}(u)/x\}\mathtt{x}(t') && \text{(by IH)} \\
&\equiv {}^{\mathtt{x}}\{\mathtt{x}(u)/x\}\langle z\mathtt{x}(s)/y\rangle\mathtt{x}(t') \\
&\equiv {}^{\mathtt{x}}\{\mathtt{x}(u)/x\}\mathtt{x}(\langle zs/y\rangle t')
\end{aligned}$$

(e) $t \equiv \langle xs/y\rangle t'$. Then

$$\begin{aligned}
&\mathtt{x}(\{u/x\}\langle xs/y\rangle t') \\
&\equiv\ \mathtt{x}([u/x]\langle \underline{x}\{u/x\}s/y\rangle\{u/x\}t') \\
&\equiv\ \mathtt{x}([\mathtt{x}(u)/x]\langle \underline{x}\mathtt{x}(\{u/x\}s)/y\rangle\mathtt{x}(\{u/x\}t')) \\
&\equiv\ \mathtt{x}([\mathtt{x}(u)/x]\langle \underline{x}\mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(s))/y\rangle\mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(t'))) && \text{(by IH)} \\
&\equiv\ \mathtt{x}([\mathtt{x}(u)/x]\langle \underline{x}\{\mathtt{x}(u)/x\}\mathtt{x}(s)/y\rangle\{\mathtt{x}(u)/x\}\mathtt{x}(t')) \\
&\equiv\ \mathtt{x}(\{\mathtt{x}(u)/x\}\langle x\mathtt{x}(s)/y\rangle\mathtt{x}(t')) \\
&\equiv\ \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(\langle xs/y\rangle t'))
\end{aligned}$$

(f) $t \equiv [s/y]t'$. We assume $y \notin FV(\mathtt{x}(u))$. Then

$$\begin{aligned}
&\mathtt{x}(\{u/x\}[s/y]t') \\
&\equiv\ \mathtt{x}([\{u/x\}s/y]\{u/x\}t') \\
&\equiv\ \mathtt{x}([\mathtt{x}(\{u/x\}s)/y]\mathtt{x}(\{u/x\}t')) \\
&\equiv\ \mathtt{x}([\mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(s))/y]\mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(t'))) && \text{(by IH)} \\
&\equiv\ \mathtt{x}(\{\mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(s))/y\}\mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(t'))) && \text{(by Lemma 3 (1))} \\
&\equiv\ \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(\{\mathtt{x}(s)/y\}\mathtt{x}(t'))) && \text{(by Lemma 8)} \\
&\equiv\ \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}([\mathtt{x}(s)/y]\mathtt{x}(t'))) && \text{(by Lemma 3 (1))} \\
&\equiv\ \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}([s/y]t'))
\end{aligned}$$

□

Lemma 15. *If* $t \to_{\text{gcut}} t'$ *then* $\mathtt{x}(t) \xrightarrow{*}_\beta \mathtt{x}(t')$.

Proof. By induction on the reduction relation $\to_{\text{gcut}}$. We treat some cases.

(a) $t \equiv [\lambda z.r/x]\langle \underline{x}s/y\rangle t_0 \to_{Beta} [[s/z]r/y]t_0 \equiv t'$. Then

$$\begin{aligned}
\mathtt{x}([\lambda z.r/x]\langle \underline{x}s/y\rangle t_0) &\equiv \mathtt{x}([\lambda z.\mathtt{x}(r)/x]\langle \underline{x}\mathtt{x}(s)/y\rangle\mathtt{x}(t_0)) \\
&\equiv [\lambda z.\mathtt{x}(r)/x]\langle \underline{x}\mathtt{x}(s)/y\rangle\mathtt{x}(t_0) \\
&\to_\beta \mathtt{x}(\{\mathtt{x}(\{\mathtt{x}(s)/z\}\mathtt{x}(r))/y\}\mathtt{x}(t_0)) \\
&\equiv \mathtt{x}([\mathtt{x}([\mathtt{x}(s)/z]\mathtt{x}(r))/y]\mathtt{x}(t_0)) && \text{(by Lemma 3 (1))} \\
&\equiv \mathtt{x}([[s/z]r/y]t_0)
\end{aligned}$$

(b) $t \equiv [u/x]\langle \underline{x}s/y\rangle t_0 \to_{left} \langle\boldsymbol{\{}u\boldsymbol{\}}s/y\rangle t_0 \equiv t'$. Then

$$\begin{aligned}
\mathtt{x}([u/x]\langle \underline{x}s/y\rangle t_0) &\equiv \mathtt{x}([\mathtt{x}(u)/x]\langle \underline{x}\mathtt{x}(s)/y\rangle\mathtt{x}(t_0)) \\
&\equiv \langle\boldsymbol{\{}\mathtt{x}(u)\boldsymbol{\}}\mathtt{x}(s)/y\rangle\mathtt{x}(t_0) && \text{(by Lemma 3 (2))} \\
&\equiv \mathtt{x}(\langle\boldsymbol{\{}u\boldsymbol{\}}s/y\rangle t_0) && \text{(by Lemma 14 (1))}
\end{aligned}$$

(c) $t \equiv [u/x]r \to_{right} \{u/x\}r \equiv t'$. Then

$$\begin{aligned}
\mathtt{x}([u/x]r) &\equiv \mathtt{x}([\mathtt{x}(u)/x]\mathtt{x}(r)) \\
&\equiv \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(r)) && \text{(by Lemma 3 (1))} \\
&\equiv \mathtt{x}(\{u/x\}r) && \text{(by Lemma 14 (2))}
\end{aligned}$$

(d) $t \equiv [u/x]r$ and $u \rightarrow_{\mathrm{gcut}} u'$. By the induction hypothesis, $\mathtt{x}(u) \xrightarrow{*}_{\beta} \mathtt{x}(u')$. Hence

$$\begin{aligned}
\mathtt{x}([u/x]r) &\equiv \mathtt{x}([\mathtt{x}(u)/x]\mathtt{x}(r)) \\
&\equiv \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(r)) && \text{(by Lemma 3 (1))} \\
&\xrightarrow{*}_{\beta} \mathtt{x}(\{\mathtt{x}(u')/x\}\mathtt{x}(r)) && \text{(by Lemma 13 (1))} \\
&\equiv \mathtt{x}([\mathtt{x}(u')/x]\mathtt{x}(r)) && \text{(by Lemma 3 (1))} \\
&\equiv \mathtt{x}([u'/x]r)
\end{aligned}$$

(e) $t \equiv [u/x]r$ and $r \rightarrow_{\mathrm{gcut}} r'$. By the induction hypothesis, $\mathtt{x}(r) \xrightarrow{*}_{\beta} \mathtt{x}(r')$. Hence

$$\begin{aligned}
\mathtt{x}([u/x]r) &\equiv \mathtt{x}([\mathtt{x}(u)/x]\mathtt{x}(r)) \\
&\equiv \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(r)) && \text{(by Lemma 3 (1))} \\
&\xrightarrow{*}_{\beta} \mathtt{x}(\{\mathtt{x}(u)/x\}\mathtt{x}(r')) && \text{(by Lemma 13 (2))} \\
&\equiv \mathtt{x}([\mathtt{x}(u)/x]\mathtt{x}(r')) && \text{(by Lemma 3 (1))} \\
&\equiv \mathtt{x}([u/x]r')
\end{aligned}$$

□