A Translation of Intersection and Union Types for the $\lambda\mu$-Calculus

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Abstract. We introduce an intersection and union type system for the $\lambda\mu$-calculus, which includes a restricted version of the traditional union-elimination rule. We give a translation from intersection and union types into intersection and product types, which is a variant of negative translation from classical logic to intuitionistic logic and naturally reflects the structure of strict intersection and union types. It is shown that a derivation in our type system can be translated into a derivation in the type system of van Bakel, Barbanera and de'Liguoro. As a corollary, the terms typable in our system turn out to be strongly normalising. We also present an intersection and union type system in the style of sequent calculus, and show that the terms typable in the system coincide with the strongly normalising terms of the $\overline{\lambda}\mu$-calculus, a call-by-name fragment of Curien and Herbelin's $\overline{\lambda}\mu\tilde{\mu}$-calculus.

1 Introduction

Since Griffin's seminal work [Gri90], the Curry-Howard correspondence for classical logic has been extensively studied and has yielded various term systems, e.g. the calculi in [BB96,Par92,CH00]. Some of those systems can be considered as calculi with control operators which deal with first-class continuations. Parigot's $\lambda\mu$-calculus [Par92] is one of such systems, and since it is a syntactical extension of the usual λ-calculus, the type-free version of the calculus, called pure $\lambda\mu$-calculus in [Par92], has also been studied.

As a type assignment system for type-free $\lambda\mu$-terms, van Bakel, Barbanera and de'Liguoro [vBBdL11] recently introduced an intersection type system to develop model theory of the calculus. The system includes not only intersection types but also product types, and so looks involved at first sight. However, the system can be naturally understood in the light of the negative translation used in [SR98], and indeed the simply typed part of the $\lambda\mu$-calculus is interpreted by the systems with intersection and product types in [vBBdL11,vBBdL13].

Another approach to providing a type assignment system for type-free $\lambda\mu$-terms is to employ a system with intersection and union types. In this approach, simple types inhabited by some terms correspond to implicational formulas that are provable in classical logic, and union types are used for continuations to have more than one type. There are two intersection and union type systems for the $\lambda\mu$-calculus in the literature [Lau04,vB11]. In this paper we introduce another intersection and union type system where, unlike in the systems of [Lau04,vB11], union-introduction and elimination rules correspond to the usual or-introduction and elimination rules in natural deduction. It is well-known in the context of λ-calculus that the presence of such a standard union-elimination rule causes difficulties for the subject reduction property (cf. [BDdL95]). So we impose some restrictions on terms in the premisses of the union-elimination rule, expecting that the system enjoys the subject reduction property (though in this paper we focus on translations between systems and leave a proof of it for future work).

To clarify the relation between the two kinds of type systems, we introduce a translation from intersection and union types to intersection and product types. As in the previous systems [Lau04,vB11], the types occurring in our system are the strict version of intersection and union types (cf. [vB92,vB11b]). Our translation is defined along the structure of this class of types. Using the translation, we show that each derivation in our system can be transformed into a derivation in the system of [vBBdL13]. This implies strong normalisation of terms typable in our system. Since strong normalisation has not been treated in [Lau04,vB11], this is a new result on intersection and union type systems for the $\lambda\mu$-calculus.

In the latter half of the paper, we introduce and study an intersection and union type system for a call-by-name fragment of Curien and Herbelin's $\overline{\lambda}\mu\tilde{\mu}$-calculus [CH00]. It is shown that the system enjoys both the subject reduction property and the characterisation of strong normalisation by means of typability. To prove one direction of the characterisation, we use the strong normalisation result mentioned above, together with a transformation from derivations in the type system into derivations in the above type system for the $\lambda\mu$-calculus and simulation of each reduction step in the calculus by at least one reduction step in the $\lambda\mu$-calculus. The transformation of derivations clarifies the relation between the type systems (in particular, the left-union and the union-elimination rules).

One of the reasons to study systems based on sequent calculus, such as the $\overline{\lambda}\mu\tilde{\mu}$-calculus, is that they embody both logical and computational duality more explicitly than systems based on natural deduction. Since union types are thought to be dual to intersection types, a system with both types was proposed in [DGL04,DGL05] for the $\overline{\lambda}\mu\tilde{\mu}$-calculus. The system employs *definite* types in type environments whereas our system uses strict types similarly to the systems for the $\lambda\mu$-calculus. Restricting types to definite ones is, however, not enough to satisfy the subject reduction property, even in the case of call-by-name or call-by-value reduction, as pointed out in [DGL08,vB10]. The system in [DGL08], which uses intersection types and an involution operator rather than union types, does not satisfy subject reduction either, as illustrated in Section 8 of [vB10].

To recover the subject reduction property of an intersection and union type system for the $\overline{\lambda}\mu\tilde{\mu}$-calculus, another restriction is required. A crucial restriction is given in Definition 23(i) of [vB10] for the case of call-by-name reduction, which is also thought to be dual to the value restriction in call-by-value functional languages (for the second-order quantification case). Since the calculus we study in this paper does not have the $\tilde{\mu}$-operator, the restriction is automatically satisfied. Our result seems to be the first characterisation of strong normalisation of (a fragment of) the $\overline{\lambda}\mu\tilde{\mu}$-calculus by means of typability in an intersection and union type system that enjoys the subject reduction property.

The organisation of the paper is as follows. In Section 2 we discuss type systems for the $\lambda\mu$-calculus and their relationships. In Section 3 we introduce a type system for a call-by-name fragment of the $\overline{\lambda}\mu\tilde{\mu}$-calculus and study its properties. In Section 4 we conclude and give suggestions for further work.

2 Intersection and union types for the $\lambda\mu$-calculus

In this section we introduce a new intersection and union type system for the $\lambda\mu$-calculus, and discuss the relationships to systems in previous work.

2.1 The $\lambda\mu$-calculus

First we introduce the syntax of Parigot's pure $\lambda\mu$-calculus [Par92].

Definition 1 (Grammar of $\lambda\mu$). *The sets of* terms *and* commands *are defined inductively by the following grammar:*

$$\begin{array}{rll} M, N ::= & x \mid \lambda x.M \mid MN \mid \mu\alpha.C & \textit{(terms)} \\ C ::= & [\alpha]M & \textit{(commands)} \end{array}$$

where x and α range over denumerable sets of λ-variables and μ-variables, respectively.

The notions of free and bound variables are defined as usual, with both λ and μ as binders. The sets of free λ-variables and μ-variables of a term M are denoted by $\mathsf{FV}_\lambda(M)$ and $\mathsf{FV}_\mu(M)$, respectively. We identify α-convertible terms and use $\equiv$ to denote syntactic equality modulo α-conversion.

Definition 2 (Reduction System of $\lambda\mu$). *The reduction rules are:*

$$\begin{array}{ll} (\beta) & (\lambda x.M)N \to M[x := N] \\ (\mu) & (\mu\alpha.C)N \to \mu\alpha.C[\alpha \Leftarrow N] \end{array}$$

where $[x := N]$ is usual capture-free substitution, and $[\alpha \Leftarrow N]$ in the rule (μ) replaces inductively each occurrence in C of the form $[\alpha]P$ by $[\alpha](PN)$.

The reduction relation $\longrightarrow_{\beta,\mu}$ is defined by the contextual closure of the rules (β) and (μ). We use $\longrightarrow^+_{\beta,\mu}$ for its transitive closure, and $\longrightarrow^*_{\beta,\mu}$ for its reflexive transitive closure. A term M is said to be *strongly normalising* if there is no infinite β,μ-reduction sequence out of M. The set of strongly normalising terms is denoted by $\mathsf{SN}^{\beta,\mu}$. These kinds of notations are also used for the notions of other reductions in this paper.

2.2 An intersection and union type system for the $\lambda\mu$-calculus

In this subsection we introduce an intersection and union type system for the $\lambda\mu$-calculus. The types we consider here can be seen as an extension of strict intersection types [vB92,vB11b]. We distinguish three kinds of types, following the definition in [Lau04] (without the empty intersection).

Definition 3. *The sets $\mathcal{T}_A, \mathcal{T}_I$ and $\mathcal{T}_U$ of three kinds of* types *are defined inductively by the following grammar:*

$$\begin{array}{lll} \mathcal{T}_A: & A, B ::= \varphi \mid I \to U & \textit{(arrow types)} \\ \mathcal{T}_I: & I, J ::= U \mid I \cap J & \textit{(intersection types)} \\ \mathcal{T}_U: & U, V ::= A \mid U \cup V & \textit{(union types)} \end{array}$$

where φ ranges over a denumerable set of type variables. We identify types modulo associativity and commutativity of $\cap$ and $\cup$, and use $\equiv$ to denote the equivalence.

The type assignment system $\lambda\mu_{\cap\cup}$ is defined by the rules in Figure 1. A *type environment*, ranged over by Γ, is defined as a finite set of pairs $\{x_1 : I_1, \ldots, x_n : I_n\}$ where the λ-variables are pairwise distinct. The type environment $\Gamma, x : I$ denotes the union $\Gamma \cup \{x : I\}$ where x does not appear in Γ. Similarly for type environments with μ-variables $\{\alpha_1 : U_1, \ldots, \alpha_n : U_m\}$, ranged over by Δ, except that the types are restricted to union types. We write $\Gamma \vdash_{\cap\cup} M : I \mid \Delta$ if $\Gamma \vdash M : I \mid \Delta$ is derivable with the rules in Figure 1.

$$\frac{}{\Gamma, x : I \vdash x : I \mid \Delta}\ (\mathsf{Ax}) \qquad \frac{\Gamma \vdash M : U \mid \alpha : U, \Delta}{\Gamma \vdash \mu\alpha.[\alpha]M : U \mid \Delta}\ (\mu_1) \qquad \frac{\Gamma \vdash M : V \mid \alpha : U, \gamma : V, \Delta}{\Gamma \vdash \mu\alpha.[\gamma]M : U \mid \gamma : V, \Delta}\ (\mu_2)$$

$$\frac{\Gamma, x : I \vdash M : U \mid \Delta}{\Gamma \vdash \lambda x.M : I \to U \mid \Delta}\ (\to \mathsf{I}) \qquad \frac{\Gamma \vdash M : I \to U \mid \Delta \quad \Gamma \vdash N : I \mid \Delta}{\Gamma \vdash MN : U \mid \Delta}\ (\to \mathsf{E})$$

$$\frac{\Gamma \vdash M : I \mid \Delta \quad \Gamma \vdash M : J \mid \Delta}{\Gamma \vdash M : I \cap J \mid \Delta}\ (\cap \mathsf{I}) \qquad \frac{\Gamma \vdash M : I \cap J \mid \Delta}{\Gamma \vdash M : I \mid \Delta}\ (\cap \mathsf{E}) \qquad \frac{\Gamma \vdash M : I \cap J \mid \Delta}{\Gamma \vdash M : J \mid \Delta}\ (\cap \mathsf{E})$$

$$\frac{\Gamma \vdash M : U \mid \Delta}{\Gamma \vdash M : U \cup V \mid \Delta}\ (\cup \mathsf{I}) \qquad \frac{\Gamma \vdash M : V \mid \Delta}{\Gamma \vdash M : U \cup V \mid \Delta}\ (\cup \mathsf{I})$$

$$\frac{\Gamma \vdash M : U \cup V \mid \Delta \quad \Gamma, x : U \vdash xN : I \mid \Delta \quad \Gamma, x : V \vdash xN : I \mid \Delta}{\Gamma \vdash MN : I \mid \Delta}\ (\cup \mathsf{E})$$

where $x \notin \mathsf{FV}_\lambda(N)$

Fig. 1. Type assignment system $\lambda\mu_{\cap\cup}$

The rule $(\cup \mathsf{E})$ in Figure 1 is a rather restricted version of the traditional union-elimination rule that would have the form:

$$\frac{\Gamma \vdash M : U \cup V \mid \Delta \quad \Gamma, x : U \vdash N : I \mid \Delta \quad \Gamma, x : V \vdash N : I \mid \Delta}{\Gamma \vdash N[x := M] : I \mid \Delta}$$

This general version causes the subject-reduction problem as in the case of an intersection and union type system for λ-terms (cf. [BDdL95]). Though our rule $(\cup\,\mathsf{E})$ might look too restrictive, the system $\lambda\mu_{\cap\cup}$ is more general than the intersection and union type systems proposed in [Lau04] and [vB11] in the sense that if a term is typable in one of their systems without the empty intersection then it is typable in $\lambda\mu_{\cap\cup}$ (cf. Subsection 2.4). An example of a judgement that is derivable in $\lambda\mu_{\cap\cup}$ but not in the systems in [Lau04,vB11] is $x : \varphi_1 \vdash x : \varphi_1 \cup \varphi_2 \mid$.

Example 1. The term $(\mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y))z$ is typable in the system $\lambda\mu_{\cap\cup}$ as follows. Let $A \equiv \varphi_1 \to \varphi_2$ and $\Gamma = \{z : \varphi_1 \cap A\}$, and let D_1 be the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, y : A \vdash y : A \mid \gamma : \varphi_3, \alpha : A \cup (A \to \varphi_3)}\,(\mathsf{Ax})}{\Gamma, y : A \vdash y : A \cup (A \to \varphi_3) \mid \gamma : \varphi_3, \alpha : A \cup (A \to \varphi_3)}\,(\cup\,\mathsf{I})}{\Gamma, y : A \vdash \mu\gamma.[\alpha]y : \varphi_3 \mid \alpha : A \cup (A \to \varphi_3)}\,(\mu_2)}{\Gamma \vdash \lambda y.\mu\gamma.[\alpha]y : A \to \varphi_3 \mid \alpha : A \cup (A \to \varphi_3)}\,(\to\mathsf{I})}{\Gamma \vdash \lambda y.\mu\gamma.[\alpha]y : A \cup (A \to \varphi_3) \mid \alpha : A \cup (A \to \varphi_3)}\,(\cup\,\mathsf{I})}{\Gamma \vdash \mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y) : A \cup (A \to \varphi_3) \mid}\,(\mu_1)$$

Let D_2 be the following derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x : A \vdash x : A \mid}\,(\mathsf{Ax}) \quad \dfrac{\dfrac{}{\Gamma, x : A \vdash z : \varphi_1 \cap A \mid}\,(\mathsf{Ax})}{\Gamma, x : A \vdash z : \varphi_1 \mid}\,(\cap\,\mathsf{E})}{\Gamma, x : A \vdash xz : \varphi_2 \mid}\,(\to\mathsf{E})}{\Gamma, x : A \vdash xz : \varphi_2 \cup \varphi_3 \mid}\,(\cup\,\mathsf{I})$$

Let D_3 be the following derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x : A \to \varphi_3 \vdash x : A \to \varphi_3 \mid}\,(\mathsf{Ax}) \quad \dfrac{\dfrac{}{\Gamma, x : A \to \varphi_3 \vdash z : \varphi_1 \cap A \mid}\,(\mathsf{Ax})}{\Gamma, x : A \to \varphi_3 \vdash z : A \mid}\,(\cap\,\mathsf{E})}{\Gamma, x : A \to \varphi_3 \vdash xz : \varphi_3 \mid}\,(\to\mathsf{E})}{\Gamma, x : A \to \varphi_3 \vdash xz : \varphi_2 \cup \varphi_3 \mid}\,(\cup\,\mathsf{I})$$

Then by applying the rule $(\cup\,\mathsf{E})$ to the conclusions of D_1, D_2 and D_3, we obtain a derivation of $\Gamma \vdash (\mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y))z : \varphi_2 \cup \varphi_3 \mid$. Note that this term is not typable without using the rules for $\cup$. □

Lemma 1.

1. *If $\Gamma \vdash_{\cap\cup} t : I \mid \Delta$ and x is a fresh λ-variable then $\Gamma, x : J \vdash_{\cap\cup} t : I \mid \Delta$.*
2. *If $\Gamma \vdash_{\cap\cup} t : I \mid \Delta$ and α is a fresh μ-variable then $\Gamma \vdash_{\cap\cup} t : I \mid \alpha : U, \Delta$.*

Proof. By induction on the derivations. □

Lemma 2. *The following rule is admissible in the system $\lambda\mu_{\cap\cup}$.*

$$\dfrac{\Gamma \vdash M : U_1 \cup \cdots \cup U_n \mid \Delta \quad \Gamma, x : U_1 \vdash xN : I \mid \Delta \quad \cdots \quad \Gamma, x : U_n \vdash xN : I \mid \Delta}{\Gamma \vdash MN : I \mid \Delta}\,(\cup\,\mathsf{E})^n$$

where $n \geq 2$ and $x \notin \mathsf{FV}_\lambda(N)$.

Proof. By induction on n. If $n > 2$ then we have, by the induction hypothesis,

$$\dfrac{\dfrac{}{\Gamma' \vdash x : U_1 \cup \cdots \cup U_{n-1} \mid \Delta}\,(\mathsf{Ax}) \quad \begin{matrix} \vdots\, D_1 \\ \Gamma', x' : U_1 \vdash x'N : I \mid \Delta \end{matrix} \quad \cdots \quad \begin{matrix} \vdots\, D_{n-1} \\ \Gamma', x' : U_{n-1} \vdash x'N : I \mid \Delta \end{matrix}}{\Gamma' \vdash xN : I \mid \Delta}\,(\cup\mathsf{E})^{n-1}$$

where $\Gamma' = \Gamma, x : U_1 \cup \cdots \cup U_{n-1}$ and each D_i is obtained from the assumption by renaming x to x'. Hence, by the rule $(\cup\mathsf{E})$, we can derive $\Gamma \vdash MN : I \mid \Delta$ from $\Gamma \vdash M : U_1 \cup \cdots \cup U_n \mid \Delta$, $\Gamma' \vdash xN : I \mid \Delta$ and $\Gamma, x : U_n \vdash xN : I \mid \Delta$. □

2.3 The type system of van Bakel, Barbanera and de'Liguoro

In this subsection we briefly recall the intersection type system in [vBBdL13] which also uses product types. The system is a modification of the system in [vBBdL11], which was inspired by denotational semantics developed in [SR98].

The intersection-free part of the system can be seen as the image of negative translation from the implication fragment of classical logic into the conjunction and negation fragment of intuitionistic logic (viewing $\times$ as conjunction and $\to \nu$ as negation). However, unlike in a CPS-translation, only types are translated in the image while terms are not changed from those in the $\lambda\mu$-calculus.

Definition 4. *The sets* $\mathcal{T}_D$ *of* term types *and* $\mathcal{T}_C$ *of* continuation-stack types *are defined inductively by the following grammar:*

$$\begin{array}{ll} \mathcal{T}_D : & \delta ::= \nu \mid \omega \to \nu \mid \kappa \to \nu \mid \delta \wedge \delta \\ \mathcal{T}_C : & \kappa ::= \delta \times \omega \mid \delta \times \kappa \mid \kappa \wedge \kappa \end{array}$$

where ν *and* ω *are type constants. Elements of* $\mathcal{T}_D \cup \mathcal{T}_C$ *are ranged over by* σ, τ, ρ.

The relations $\leq_D$ and $\leq_C$ on $\mathcal{T}_D$ and $\mathcal{T}_C$, respectively, are defined by the rules in Figure 2, where $\leq_A$ denotes either $\leq_D$ or $\leq_C$.

$$\dfrac{}{\sigma \leq_A \sigma} \quad \dfrac{}{\sigma \wedge \tau \leq_A \sigma} \quad \dfrac{}{\sigma \wedge \tau \leq_A \tau} \quad \dfrac{}{\nu \leq_D \omega \to \nu} \quad \dfrac{}{\omega \to \nu \leq_D \nu} \quad \dfrac{}{\delta_1 \times \delta_2 \times \omega \leq_C \delta_1 \times \omega}$$

$$\dfrac{}{(\delta_1 \times \omega) \wedge (\delta_2 \times \kappa) \leq_C (\delta_1 \wedge \delta_2) \times \kappa} \quad \dfrac{}{(\delta_1 \times \kappa_1) \wedge (\delta_2 \times \kappa_2) \leq_C (\delta_1 \wedge \delta_2) \times (\kappa_1 \wedge \kappa_2)}\ \text{where } \kappa_1 \neq \omega \text{ and } \kappa_2 \neq \omega$$

$$\dfrac{\sigma \leq_A \rho \quad \rho \leq_A \tau}{\sigma \leq_A \tau} \quad \dfrac{\sigma \leq_A \tau_1 \quad \sigma \leq_A \tau_2}{\sigma \leq_A \tau_1 \wedge \tau_2}$$

$$\dfrac{\delta_1 \leq_D \delta_2}{\delta_1 \times \omega \leq_C \delta_2 \times \omega} \quad \dfrac{\delta_1 \leq_D \delta_2 \quad \kappa_1 \leq_C \kappa_2}{\delta_1 \times \kappa_1 \leq_C \delta_2 \times \kappa_2} \quad \dfrac{\kappa_2 \leq_C \kappa_1}{\kappa_1 \to \nu \leq_D \kappa_2 \to \nu}$$

Fig. 2. Relations $\leq_D$ and $\leq_C$

Lemma 3. $(\delta_1 \wedge \cdots \wedge \delta_n) \times (\kappa_1 \wedge \cdots \wedge \kappa_n) \leq_C (\delta_1 \times \kappa_1) \wedge \cdots \wedge (\delta_n \times \kappa_n)$.

Proof. By $\delta_1 \wedge \cdots \wedge \delta_n \leq_D \delta_i$ and $\kappa_1 \wedge \cdots \wedge \kappa_n \leq_C \kappa_i$, we have $(\delta_1 \wedge \cdots \wedge \delta_n) \times (\kappa_1 \wedge \cdots \wedge \kappa_n) \leq_C \delta_i \times \kappa_i$ for each $i \in \{1, \ldots, n\}$. Hence $(\delta_1 \wedge \cdots \wedge \delta_n) \times (\kappa_1 \wedge \cdots \wedge \kappa_n) \leq_C (\delta_1 \times \kappa_1) \wedge \cdots \wedge (\delta_n \times \kappa_n)$. □

The type assignment system $\lambda\mu_{\wedge\times}$ is defined by the rules in Figure 3. We write $\Gamma \vdash_{\wedge\times} M : \delta \mid \Delta$ if $\Gamma \vdash M : \delta \mid \Delta$ is derivable with the rules of Figure 3.

$$\frac{}{\Gamma, x : \delta \vdash x : \delta \mid \Delta}\ (\mathsf{Ax}) \qquad \frac{\Gamma \vdash M : \kappa \to \nu \mid \alpha : \kappa, \Delta}{\Gamma \vdash \mu\alpha.[\alpha]M : \kappa \to \nu \mid \Delta}\ (\mu^\nu_1) \qquad \frac{\Gamma \vdash M : \kappa' \to \nu \mid \alpha : \kappa, \gamma : \kappa', \Delta}{\Gamma \vdash \mu\alpha.[\gamma]M : \kappa \to \nu \mid \gamma : \kappa', \Delta}\ (\mu^\nu_2)$$

$$\frac{\Gamma, x : \delta \vdash M : \kappa \to \nu \mid \Delta}{\Gamma \vdash \lambda x.M : \delta \times \kappa \to \nu \mid \Delta}\ (\mathsf{Abs}) \qquad \frac{\Gamma \vdash M : \delta \times \kappa \to \nu \mid \Delta \quad \Gamma \vdash N : \delta \mid \Delta}{\Gamma \vdash MN : \kappa \to \nu \mid \Delta}\ (\mathsf{App})$$

where κ is either a type in $\mathcal{T}_C$ or ω where κ is either a type in $\mathcal{T}_C$ or ω

$$\frac{\Gamma \vdash M : \delta \mid \Delta \quad \Gamma \vdash M : \delta' \mid \Delta}{\Gamma \vdash M : \delta \wedge \delta' \mid \Delta}\ (\wedge) \qquad \frac{\Gamma \vdash M : \delta \mid \Delta \quad \delta \leq_D \delta'}{\Gamma \vdash M : \delta' \mid \Delta}\ (\leq)$$

Fig. 3. Type assignment system $\lambda\mu_{\wedge\times}$

In [vBBdL13], it was shown that strongly normalising terms are characterised by means of typability in the system $\lambda\mu_{\wedge\times}$. The following is one direction of the characterisation theorem.

Theorem 1 ([vBBdL13]). *If $\Gamma \vdash_{\wedge\times} M : \delta \mid \Delta$ for some Γ, δ and Δ, then $M \in \mathsf{SN}^{\beta,\mu}$.*

2.4 A translation of intersection and union types

Now we introduce a translation of intersection and union types into intersection and product types, extending the translation of simple types in [vBBdL13]. The translation is defined along the structure of the three kinds of types in Definition 3. The aim is to prove strong normalisation of terms typable in $\lambda\mu_{\cap\cup}$, using this translation and the strong normalisation result of terms typable in $\lambda\mu_{\wedge\times}$.

Definition 5. *The mappings $(\cdot)^D : \mathcal{T}_I \to \mathcal{T}_D$ and $(\cdot)^C : \mathcal{T}_U \to \mathcal{T}_C$ are defined inductively as follows:*

$$\begin{aligned} \varphi^C &:= \nu \times \omega \\ (I \to U)^C &:= I^D \times U^C \\ (U \cup V)^C &:= U^C \wedge V^C \\[4pt] U^D &:= U^C \to \nu \\ (I \cap J)^D &:= I^D \wedge J^D \end{aligned}$$

It can be easily verified that the above mappings are well-defined. We extend the mappings to type environments by $\Gamma^D := \{x : I^D \mid x : I \in \Gamma\}$ and $\Delta^C := \{\alpha : U^C \mid \alpha : U \in \Delta\}$.

Proving the preservation of derivability in $\lambda\mu_{\cap\cup}$ by the translation requires some observations on the system $\lambda\mu_{\wedge\times}$. We give a detailed proof of it in Appendix A. Here we instead show the preservation of derivability in the system that is obtained from $\lambda\mu_{\cap\cup}$ by replacing the rules $(\to \mathsf{E})$ and $(\cup \mathsf{E})$ with the following one:

$$\frac{\Gamma \vdash M : (I_1 \to U_1) \cup \cdots \cup (I_n \to U_n) \mid \Delta \quad \Gamma \vdash N : I_1 \mid \Delta \quad \cdots \quad \Gamma \vdash N : I_n \mid \Delta}{\Gamma \vdash MN : U_1 \cup \cdots \cup U_n \mid \Delta} \ (\to \mathsf{E})'$$

where $n \geq 1$. This rule is the same as one of the rules of the intersection and union type system in [vB11]. A similar rule also appeared in [Lau04]. We write $\Gamma \vdash_{\cap\cup'} M : \delta \mid \Delta$ if $\Gamma \vdash M : \delta \mid \Delta$ is derivable in this alternative system. By using the rule $(\cup \mathsf{E})^n$ in Lemma 2, we see that the rule $(\to \mathsf{E})'$ is derivable in the original system $\lambda\mu_{\cap\cup}$, so $\Gamma \vdash_{\cap\cup'} M : \delta \mid \Delta$ implies $\Gamma \vdash_{\cap\cup} M : \delta \mid \Delta$.

Theorem 2. *If $\Gamma \vdash_{\cap\cup'} M : I \mid \Delta$ then $\Gamma^D \vdash_{\wedge\times} M : I^D \mid \Delta^C$.*

Proof. By induction on the derivation of $\Gamma \vdash_{\cap\cup'} M : I \mid \Delta$. Here we consider some cases.

- $\dfrac{\Gamma \vdash M : V \mid \alpha : U, \gamma : V, \Delta}{\Gamma \vdash \mu\alpha.[\gamma]M : U \mid \gamma : V, \Delta} \ (\mu_2)$

  By the induction hypothesis, we have $\Gamma^D \vdash_{\wedge\times} M : V^D \mid \alpha : U^C, \gamma : V^C, \Delta^C$ where $V^D \equiv V^C \to \nu$. Then by the rule (μ_2^ν), we obtain $\Gamma^D \vdash_{\wedge\times} \mu\alpha.[\gamma]M : U^C \to \nu \mid \gamma : V^C, \Delta^C$.

- $\dfrac{\Gamma, x : I \vdash M : U \mid \Delta}{\Gamma \vdash \lambda x.M : I \to U \mid \Delta} \ (\to \mathsf{I})$

  By the induction hypothesis, we have $\Gamma^D, x : I^D \vdash_{\wedge\times} M : U^D \mid \Delta^C$ where $U^D \equiv U^C \to \nu$. Then by the rule (Abs), we obtain $\Gamma^D \vdash_{\wedge\times} \lambda x.M : I^D \times U^C \to \nu \mid \Delta^C$ where $I^D \times U^C \to \nu \equiv (I \to U)^C \to \nu \equiv (I \to U)^D$.

- $\dfrac{\Gamma \vdash M : U \mid \Delta}{\Gamma \vdash M : U \cup V \mid \Delta} \ (\cup \mathsf{I})$

  By the induction hypothesis, we have $\Gamma^D \vdash_{\wedge\times} M : U^D \mid \Delta^C$ where $U^D \equiv U^C \to \nu$. From the definition of $\leq_D$, we have $U^C \to \nu \leq_D U^C \wedge V^C \to \nu$. Hence by the rule $(\leq)$, we obtain $\Gamma^D \vdash_{\wedge\times} M : U^C \wedge V^C \to \nu \mid \Delta^C$ where $U^C \wedge V^C \to \nu \equiv (U \cup V)^C \to \nu \equiv (U \cup V)^D$.

- $\dfrac{\Gamma \vdash M : (I_1 \to U_1) \cup \cdots \cup (I_n \to U_n) \mid \Delta \quad \Gamma \vdash N : I_1 \mid \Delta \quad \cdots \quad \Gamma \vdash N : I_n \mid \Delta}{\Gamma \vdash MN : U_1 \cup \cdots \cup U_n \mid \Delta} \ (\to \mathsf{E})'$

  By the induction hypothesis, we have $\Gamma^D \vdash_{\wedge\times} M : ((I_1 \to U_1) \cup \cdots \cup (I_n \to U_n))^D \mid \Delta^C$ and, for all $i \in \{1, \ldots, n\}$, $\Gamma^D \vdash_{\wedge\times} N : I_i^D \mid \Delta^C$. Then by the rule $(\wedge)$, we have $\Gamma^D \vdash_{\wedge\times} N : I_1^D \wedge \cdots \wedge I_n^D \mid \Delta^C$. Now

$$\begin{aligned} ((I_1 \to U_1) \cup \cdots \cup (I_n \to U_n))^D &\equiv ((I_1 \to U_1) \cup \cdots \cup (I_n \to U_n))^C \to \nu \\ &\equiv (I_1 \to U_1)^C \wedge \cdots \wedge (I_n \to U_n)^C \to \nu \\ &\equiv (I_1^D \times U_1^C) \wedge \cdots \wedge (I_n^D \times U_n^C) \to \nu \\ &\leq_D (I_1^D \wedge \cdots \wedge I_n^D) \times (U_1^C \wedge \cdots \wedge U_n^C) \to \nu \\ &\qquad \text{(by Lemma 3)} \end{aligned}$$

Hence by the rules ($\leq$) and (App), we obtain $\Gamma^D \vdash_{\wedge\times} MN : (U_1^C \wedge \cdots \wedge U_n^C) \to \nu \mid \Delta^C$ where $(U_1^C \wedge \cdots \wedge U_n^C) \to \nu \equiv (U_1 \cup \cdots \cup U_n)^C \to \nu \equiv (U_1 \cup \cdots \cup U_n)^D$. $\square$

Corollary 1. *If $\Gamma \vdash_{\cap\cup'} M : I \mid \Delta$ for some Γ, I and Δ, then $M \in \mathsf{SN}^{\beta,\mu}$.*

Proof. By Theorems 1 and 2. $\square$

The above corollary is enough to show strong normalisation of terms typable in the systems (without the type constants) of [Lau04] and [vB11]. We can also prove the same result for the full system $\lambda\mu_{\cap\cup}$. (See Appendix A.)

Theorem 3. *If $\Gamma \vdash_{\cap\cup} M : I \mid \Delta$ then $\Gamma^D \vdash_{\wedge\times} M : I^D \mid \Delta^C$.*

Corollary 2. *If $\Gamma \vdash_{\cap\cup} M : I \mid \Delta$ for some Γ, I and Δ, then $M \in \mathsf{SN}^{\beta,\mu}$.*

3 Intersection and union types for the $\overline{\lambda}\mu$-calculus

In the remainder of the paper, we are concerned with the $\overline{\lambda}\mu$-calculus, a call-by-name fragment of Curien and Herbelin's $\overline{\lambda}\mu\tilde{\mu}$-calculus [CH00]. We present an intersection and union type system that enjoys both the subject reduction property and the characterisation of strongly normalising terms by means of typability. The strong normalisation result in the previous section is used to prove one direction of the characterisation theorem.

There are two main reasons to study here systems based on sequent calculus instead of the $\lambda\mu$-calculus based on natural deduction. One is that they embody the duality between call-by-name and call-by-value more explicitly than the $\lambda\mu$-calculus does. Though we only deal with a call-by-name calculus in the present paper, this leads to future investigations into call-by-value calculi. The other is a more technical reason. In the presence of the syntactic category of contexts, we can treat operation concerning μ-variables (i.e. $[\alpha \Leftarrow N]$ in Definition 2) as usual capture-free substitution (i.e. $[\alpha := e]$ in Definition 7). This allows us to prove properties of the systems, such as subject reduction, in a smoother way.

3.1 The $\overline{\lambda}\mu$-calculus

The $\overline{\lambda}\mu$-calculus was originally introduced in [Her95] as an extension of the $\overline{\lambda}$-calculus [Her94,Her95]. Here we study a version in [CH00] that is a call-by-name fragment of the $\overline{\lambda}\mu\tilde{\mu}$-calculus.

Definition 6 (Grammar of $\overline{\lambda}\mu$). *The sets of* terms*,* contexts *and* commands *are defined inductively by the following grammar:*

$$\begin{aligned} t, s &::= x \mid \lambda x.t \mid \mu\alpha.c && \textit{(terms)} \\ e &::= \alpha \mid t \cdot e && \textit{(contexts)} \\ c &::= \langle t \,|\, e \rangle && \textit{(commands)} \end{aligned}$$

where x and α range over denumerable sets of λ-variables and μ-variables, respectively.

The syntax has three kinds of expressions: terms, contexts and commands. Contexts are typically constructed from a μ-variable using the constructor '$\cdot$'. They can also be considered to have a hole in the position of the head variable. So the command $\langle t \,|\, e \rangle$ is read as the result of filling the hole of e with a term t.

Definition 7 (Reduction System of $\overline{\lambda}\mu$). *The reduction rules are:*

$$\begin{array}{ll} (\overline{\beta}) & \langle \lambda x.t \,|\, s \cdot e \rangle \to \langle t[x := s] \,|\, e \rangle \\ (\overline{\mu}) & \quad \langle \mu\alpha.c \,|\, e \rangle \to c[\alpha := e] \end{array}$$

where both $[x := s]$ and $[\alpha := e]$ are usual capture-free substitution.

The rule $(\overline{\beta})$ corresponds to (β) of the $\lambda\mu$-calculus, while the rule $(\overline{\mu})$ corresponds to consecutive applications of the rule (μ). A more precise correspondence is shown in Theorem 6.

3.2 An intersection and union type system for the $\overline{\lambda}\mu$-calculus

The type assignment system $\overline{\lambda}\mu_{\cap\cup}$ is defined by the rules in Figure 4. This type system is a sequent calculus based on three kinds of judgements: $\Gamma \vdash t : I \mid \Delta$, $\Gamma \mid e : I \vdash \Delta$ and $\langle t \,|\, e \rangle : (\Gamma \vdash \Delta)$. In the judgement $\Gamma \mid e : I \vdash \Delta$, the type I represents the type of the hole of the context e. So in the rule $(\mathsf{L} \to)$, the hole with type U in the right premiss is replaced, in the conclusion, by the hole with type $I \to U$ applied to the term t which is typed with I in the left premiss.

$$\frac{}{\Gamma, x : I_1 \cap \cdots \cap I_n \vdash x : I_i \mid \Delta}\ (\mathsf{Ax}) \qquad \frac{}{\Gamma \mid \alpha : U_i \vdash \alpha : U_1 \cup \cdots \cup U_n, \Delta}\ (\mathsf{Ax})$$
where $i \in \{1, \ldots, n\}$ / where $i \in \{1, \ldots, n\}$

$$\frac{\Gamma \vdash t : I \mid \Delta \quad \Gamma \mid e : I \vdash \Delta}{\langle t \,|\, e \rangle : (\Gamma \vdash \Delta)}\ (\mathsf{Cut}) \qquad \frac{c : (\Gamma \vdash \alpha : U, \Delta)}{\Gamma \vdash \mu\alpha.c : U \mid \Delta}\ (\mathsf{MuAbs})$$

$$\frac{\Gamma \vdash t : I \mid \Delta \quad \Gamma \mid e : U \vdash \Delta}{\Gamma \mid t \cdot e : I \to U \vdash \Delta}\ (\mathsf{L} \to) \qquad \frac{\Gamma, x : I \vdash t : U \mid \Delta}{\Gamma \vdash \lambda x.t : I \to U \mid \Delta}\ (\mathsf{R} \to)$$

$$\frac{\Gamma \mid e : I_i \vdash \Delta}{\Gamma \mid e : I_1 \cap I_2 \vdash \Delta}\ (\mathsf{L}\cap) \qquad \frac{\Gamma \vdash t : I \mid \Delta \quad \Gamma \vdash t : J \mid \Delta}{\Gamma \vdash t : I \cap J \mid \Delta}\ (\mathsf{R}\cap)$$
where $i \in \{1, 2\}$

$$\frac{\Gamma \mid e : U \vdash \Delta \quad \Gamma \mid e : V \vdash \Delta}{\Gamma \mid e : U \cup V \vdash \Delta}\ (\mathsf{L}\cup) \qquad \frac{\Gamma \vdash t : U_i \mid \Delta}{\Gamma \vdash t : U_1 \cup U_2 \mid \Delta}\ (\mathsf{R}\cup)$$
where $i \in \{1, 2\}$

Fig. 4. Type assignment system $\overline{\lambda}\mu_{\cap\cup}$

We write $\Gamma \vdash^{-}_{\cap\cup} t : I \mid \Delta$ (resp. $\Gamma \mid e : I \vdash^{-}_{\cap\cup} \Delta$ and $\langle t \,|\, e \rangle : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$) if $\Gamma \vdash t : I \mid \Delta$ (resp. $\Gamma \mid e : I \vdash \Delta$ and $\langle t \,|\, e \rangle : (\Gamma \vdash \Delta)$) is derivable with the rules in Figure 4.

One of the differences from the systems in [DGL04,DGL05,vB10] is that they use *definite* types in type environments while we use the types in Definition 3. Definite types are, roughly speaking, those which allow neither union types in the (immediate) components of an intersection type nor intersection types in the (immediate) components of a union type.

The next example shows that the command $\langle \mu\alpha.\langle \lambda y.\mu\gamma.\langle y \,|\, \alpha\rangle \,|\, \alpha\rangle \,|\, z \cdot \delta\rangle$ is typable in the system $\overline{\lambda}\mu_{\cap\cup}$. Through the translation in the next subsection, this command corresponds to $[\delta]((\mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y))z)$ in the $\lambda\mu$-calculus, where the subterm $(\mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y))z$ is the term treated in Example 1.

Example 2. The command $\langle \mu\alpha.\langle \lambda y.\mu\gamma.\langle y \,|\, \alpha\rangle \,|\, \alpha\rangle \,|\, z \cdot \delta\rangle$ is typable in the system $\overline{\lambda}\mu_{\cap\cup}$ as follows. Let $A \equiv \varphi_1 \to \varphi_2$, $\Gamma = \{z : \varphi_1 \cap A\}$ and $\Delta = \{\alpha : A \cup (A \to \varphi_3), \delta : \varphi_2 \cup \varphi_3\}$, and let D_1 be the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, y : A \vdash y : A \mid \gamma : \varphi_3, \Delta}(\mathsf{Ax}) \quad \dfrac{}{\Gamma, y : A \mid \alpha : A \vdash \gamma : \varphi_3, \Delta}(\mathsf{Ax})}{\langle y \,|\, \alpha\rangle : (\Gamma, y : A \vdash \gamma : \varphi_3, \Delta)}(\mathsf{Cut})}{\Gamma, y : A \vdash \mu\gamma.\langle y \,|\, \alpha\rangle : \varphi_3 \mid \Delta}(\mathsf{MuAbs})}{\Gamma \vdash \lambda y.\mu\gamma.\langle y \,|\, \alpha\rangle : A \to \varphi_3 \mid \Delta}(\to \mathsf{I}) \quad \dfrac{}{\Gamma \mid \alpha : A \to \varphi_3 \vdash \Delta}(\mathsf{Ax})}{\dfrac{\langle \lambda y.\mu\gamma.\langle y \,|\, \alpha\rangle \,|\, \alpha\rangle : (\Gamma \vdash \Delta)}{\Gamma \vdash \mu\alpha.\langle \lambda y.\mu\gamma.\langle y \,|\, \alpha\rangle \,|\, \alpha\rangle : A \cup (A \to \varphi_3) \mid \delta : \varphi_2 \cup \varphi_3}(\mathsf{MuAbs})}(\mathsf{Cut})$$

Let $\Delta' = \{\delta : \varphi_2 \cup \varphi_3\}$, and let D_2 be the following derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash z : \varphi_1 \mid \Delta'}(\mathsf{Ax}) \quad \dfrac{}{\Gamma \mid \delta : \varphi_2 \vdash \Delta'}(\mathsf{Ax})}{\Gamma \mid z \cdot \delta : A \vdash \Delta'}(\mathsf{L} \to) \quad \dfrac{\dfrac{}{\Gamma \vdash z : A \mid \Delta'}(\mathsf{Ax}) \quad \dfrac{}{\Gamma \mid \delta : \varphi_3 \vdash \Delta'}(\mathsf{Ax})}{\Gamma \mid z \cdot \delta : A \to \varphi_3 \vdash \Delta'}(\mathsf{L} \to)}{\Gamma \mid z \cdot \delta : A \cup (A \to \varphi_3) \vdash \Delta'}(\mathsf{L} \cup)$$

Then by applying the rule (Cut) to the conclusions of D_1 and D_2, we obtain a derivation of $\langle \mu\alpha.\langle \lambda y.\mu\gamma.\langle y \,|\, \alpha\rangle \,|\, \alpha\rangle \,|\, z \cdot \delta\rangle : (\Gamma \vdash \Delta')$. ◻

In the following we show some lemmas on properties of the system $\overline{\lambda}\mu_{\cap\cup}$.

Lemma 4.

1. *If* $\Gamma \vdash^-_{\cap\cup} t : I_1 \cap \cdots \cap I_n \mid \Delta$ *then* $\Gamma \vdash^-_{\cap\cup} t : I_i \mid \Delta$ *for any* $i \in \{1, \ldots, n\}$.
2. *If* $\Gamma \mid e : U_1 \cup \cdots \cup U_n \vdash^-_{\cap\cup} \Delta$ *then* $\Gamma \mid e : U_i \vdash^-_{\cap\cup} \Delta$ *for any* $i \in \{1, \ldots, n\}$.

Proof. By induction on the derivations. Note that if the last applied rule of the derivation of $\Gamma \vdash^-_{\cap\cup} t : I_1 \cap \cdots \cap I_n \mid \Delta$ is (MuAbs), (R →) or (R ∪), then $n = 1$, and that the last applied rule of the derivation of $\Gamma \mid e : U_1 \cup \cdots \cup U_n \vdash^-_{\cap\cup} \Delta$ is not (L ∩). ◻

Lemma 5 (Term Substitution Lemma). *Let* $\Gamma \vdash^-_{\cap\cup} s : I \mid \Delta$.

1. *If* $\Gamma, x : I \vdash^-_{\cap\cup} t : J \mid \Delta$ *then* $\Gamma \vdash^-_{\cap\cup} t[x := s] : J \mid \Delta$.
2. *If* $\Gamma, x : I \mid e : J \vdash^-_{\cap\cup} \Delta$ *then* $\Gamma \mid e[x := s] : J \vdash^-_{\cap\cup} \Delta$.
3. *If* $c : (\Gamma, x : I \vdash^-_{\cap\cup} \Delta)$ *then* $c[x := s] : (\Gamma \vdash^-_{\cap\cup} \Delta)$.

Proof. By simultaneous induction on the derivations. In the case where $\Gamma, x : I \vdash^{-}_{\cap\cup} t : J \mid \Delta$ is an axiom with $t \equiv x$, then we use Lemma 4(1). □

Lemma 6 (Context Substitution Lemma). *Let* $\Gamma \mid e : U \vdash^{-}_{\cap\cup} \Delta$.

1. *If* $\Gamma \vdash^{-}_{\cap\cup} t : I \mid \alpha : U, \Delta$ *then* $\Gamma \vdash^{-}_{\cap\cup} t[\alpha := e] : I \mid \Delta$.
2. *If* $\Gamma \mid e' : I \vdash^{-}_{\cap\cup} \alpha : U, \Delta$ *then* $\Gamma \mid e'[\alpha := e] : I \vdash^{-}_{\cap\cup} \Delta$.
3. *If* $c : (\Gamma \vdash^{-}_{\cap\cup} \alpha : U, \Delta)$ *then* $c[\alpha := e] : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$.

Proof. By simultaneous induction on the derivations. In the case where $\Gamma \mid e' : I \vdash^{-}_{\cap\cup} \alpha : U, \Delta$ is an axiom with $e' \equiv \alpha$, then we use Lemma 4(2). □

Lemma 7 (Generation Lemma).

1. *If* $\Gamma \mid e : I \vdash^{-}_{\cap\cup} \Delta$ *then* $I \equiv U_1 \cap \cdots \cap U_n$ *and* $\Gamma \mid e : U_i \vdash^{-}_{\cap\cup} \Delta$ *for some* U_i $(i \in \{1, \ldots, n\})$.
2. *If* $\Gamma \vdash^{-}_{\cap\cup} \lambda x.t : U \mid \Delta$ *then* $U \equiv A_1 \cup \cdots \cup A_n$ *and* $\Gamma, x : I \vdash^{-}_{\cap\cup} t : V \mid \Delta$ *for some* $A_i \equiv I \to V$ $(i \in \{1, \ldots, n\})$.
3. *If* $\Gamma \vdash^{-}_{\cap\cup} \mu\alpha.c : U \mid \Delta$ *then* $U \equiv U_1 \cup \cdots \cup U_n$ *and* $c : (\Gamma \vdash^{-}_{\cap\cup} \alpha : U_i, \Delta)$ *for some* U_i $(i \in \{1, \ldots, n\})$.
4. *If* $\Gamma \mid t \cdot e : U \vdash^{-}_{\cap\cup} \Delta$ *then* $U \equiv (I_1 \to V_1) \cup \cdots \cup (I_n \to V_n)$, $\Gamma \vdash^{-}_{\cap\cup} t : I_i \mid \Delta$ *and* $\Gamma \mid e : V_i \vdash^{-}_{\cap\cup} \Delta$ *for any* $I_i \to V_i$ $(i \in \{1, \ldots, n\})$.

Proof. By induction on the derivations. □

We are now in a position to show that the system $\overline{\lambda}\mu_{\cap\cup}$ satisfies the subject reduction property. First we prove the case where the reduction is at the root.

Lemma 8.

1. *If* $\langle \mu\alpha.c \,|\, e \rangle : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$ *then* $c[\alpha := e] : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$.
2. *If* $\langle \lambda x.t \,|\, s \cdot e \rangle : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$ *then* $\langle t[x := s] \,|\, e \rangle : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$.

Proof.
1. Let $\langle \mu\alpha.c \,|\, e \rangle : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$. Then there exists $I \equiv U_1 \cap \cdots \cap U_n$ such that $\Gamma \vdash^{-}_{\cap\cup} \mu\alpha.c : I \mid \Delta$ and $\Gamma \mid e : I \vdash^{-}_{\cap\cup} \Delta$. By Lemma 7(1), there exists U_i such that $\Gamma \mid e : U_i \vdash^{-}_{\cap\cup} \Delta$, and by Lemma 4(1), $\Gamma \vdash^{-}_{\cap\cup} \mu\alpha.c : U_i \mid \Delta$. So by Lemma 7(3), $U_i \equiv U_{i_1} \cup \cdots \cup U_{i_m}$ and $c : (\Gamma \vdash^{-}_{\cap\cup} \alpha : U_{i_k}, \Delta)$ for some U_{i_k}. Then by Lemma 4(2), $\Gamma \mid e : U_{i_k} \vdash^{-}_{\cap\cup} \Delta$. Hence by Lemma 6(3), we have $c[\alpha := e] : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$.
2. Let $\langle \lambda x.t \,|\, s \cdot e \rangle : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$. Then there exists $I \equiv U_1 \cap \cdots \cap U_n$ such that $\Gamma \vdash^{-}_{\cap\cup} \lambda x.t : I \mid \Delta$ and $\Gamma \mid s \cdot e : I \vdash^{-}_{\cap\cup} \Delta$. By Lemma 7(1), there exists U_i such that $\Gamma \mid s \cdot e : U_i \vdash^{-}_{\cap\cup} \Delta$, and by Lemma 4(1), $\Gamma \vdash^{-}_{\cap\cup} \lambda x.t : U_i \mid \Delta$. So by Lemma 7(2), $U_i \equiv A_{i_1} \cup \cdots \cup A_{i_m}$ and $\Gamma, x : J \vdash^{-}_{\cap\cup} t : V \mid \Delta$ for some $A_{i_k} \equiv J \to V$. Then by Lemma 7(4), $\Gamma \vdash^{-}_{\cap\cup} s : J \mid \Delta$ and $\Gamma \mid e : V \vdash^{-}_{\cap\cup} \Delta$. Now by Lemma 5(1), we have $\Gamma \vdash^{-}_{\cap\cup} t[x := s] : V \mid \Delta$. Hence by the rule (Cut), we obtain $\langle t[x := s] \,|\, e \rangle : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$. □

Theorem 4 (Subject Reduction).

1. *If* $\Gamma \vdash^{-}_{\cap\cup} t : I \mid \Delta$ *and* $t \longrightarrow_{\overline{\beta},\overline{\mu}} t'$ *then* $\Gamma \vdash^{-}_{\cap\cup} t' : I \mid \Delta$.
2. *If* $\Gamma \mid e : I \vdash^{-}_{\cap\cup} \Delta$ *and* $e \longrightarrow_{\overline{\beta},\overline{\mu}} e'$ *then* $\Gamma \mid e' : I \vdash^{-}_{\cap\cup} \Delta$.
3. *If* $c : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$ *and* $c \longrightarrow_{\overline{\beta},\overline{\mu}} c'$ *then* $c' : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$.

Proof. By simultaneous induction on the derivations. If the reduction is at the root, then we use Lemma 8. □

3.3 Translating $\overline{\lambda}\mu_{\cap\cup}$ into $\lambda\mu_{\cap\cup}$

In this subsection we show that typing derivations in the system $\overline{\lambda}\mu_{\cap\cup}$ can be translated into ones in the system $\lambda\mu_{\cap\cup}$ in an appropriate way. To do so, we employ an equivalent formulation to $\lambda\mu_{\cap\cup}$ that has the following rules instead of (μ_1) and (μ_2):

$$\frac{\Gamma \vdash M : U \mid \alpha : U, \Delta}{[\alpha]M : (\Gamma \vdash \alpha : U, \Delta)}\ (\mathsf{Nam}) \qquad \frac{C : (\Gamma \vdash \alpha : U, \Delta)}{\Gamma \vdash \mu\alpha.C : U \mid \Delta}\ (\mathsf{MuAbs})$$

The rule (Nam) introduces a new form of judgement $C : (\Gamma \vdash \Delta)$, which should be immediately followed by the rule (MuAbs). So the derivability of judgements of the form $\Gamma \vdash M : I \mid \Delta$ is not changed from that in the original $\lambda\mu_{\cap\cup}$. We write $C : (\Gamma \vdash_{\cap\cup} \Delta)$ if $C : (\Gamma \vdash \Delta)$ is derivable in this alternative formulation.

Also, we add the following to the reduction rules of the $\lambda\mu$-calculus:

$$(\rho) \quad [\alpha](\mu\gamma.C) \to C[\gamma := \alpha]$$

The translation from the terms in the $\overline{\lambda}\mu$-calculus to those in the $\lambda\mu$-calculus is given in Figure 5.

$$\begin{array}{rcl}
\Theta(x) & := & x \\
\Theta(\lambda x.t) & := & \lambda x.\Theta(t) \\
\Theta(\mu\alpha.c) & := & \mu\alpha.\Theta'(c) \\
\Theta'(\langle t \,|\, e\rangle) & := & \Theta''(\Theta(t), e) \\
\Theta''(M, \alpha) & := & [\alpha]M \\
\Theta''(M, t \cdot e) & := & \Theta''(M\Theta(t), e)
\end{array}$$

Fig. 5. Translation from $\overline{\lambda}\mu$ to $\lambda\mu$

The following two technical lemmas are useful for proving Theorem 5.

Lemma 9. *If $\Gamma \mid \alpha : U \vdash^{-}_{\cap\cup} \Delta$ then there exists $\alpha : V \in \Delta$ such that $V \equiv U \cup V_1 \cup \cdots \cup V_n$ $(n \geq 0)$.*

Proof. By induction on the derivation, assuming the commutativity of $\cup$. □

Lemma 10. *Let $\Gamma \mid t \cdot e : (I_1 \to U_1) \cup \cdots \cup (I_n \to U_n) \vdash^{-}_{\cap\cup} \Delta$ with its derivation length k. Then, for any $i \in \{1, \ldots, n\}$, $\Gamma \vdash^{-}_{\cap\cup} t : I_i \mid \Delta$ with its derivation length less than k, and $\Gamma \mid e : U_1 \cup \cdots \cup U_n \vdash^{-}_{\cap\cup} \Delta$ with its derivation length $k - 1$.*

Proof. As in Lemma 7(4), we have $\Gamma \vdash^{-}_{\cap\cup} t : I_i \mid \Delta$ and $\Gamma \mid e : U_i \vdash^{-}_{\cap\cup} \Delta$ for all $i \in \{1, \ldots, n\}$, whose derivations are subderivations of the given derivation. Then we can construct a derivation of $\Gamma \mid e : U_1 \cup \cdots \cup U_n \vdash \Delta$ with its length $k - 1$ by replacing each judgement for $t \cdot e$ and $(I_i \to U_i)$'s in the original derivation by a corresponding judgement for e and U_i's. □

Now we prove that the translation Θ preserves types. This explains how the rules of $\overline{\lambda}\mu_{\cap\cup}$ corresponds to the rules of $\lambda\mu_{\cap\cup}$.

Theorem 5.

1. *If* $\Gamma \vdash^{-}_{\cap\cup} t : I \mid \Delta$ *then* $\Gamma \vdash_{\cap\cup} \Theta(t) : I \mid \Delta$.
2. *If* $\Gamma \mid e : I \vdash^{-}_{\cap\cup} \Delta$ *then* $\Theta''(M, e) : (\Gamma \vdash_{\cap\cup} \Delta)$ *for any* M *such that* $\Gamma \vdash_{\cap\cup} M : I \mid \Delta$.
3. *If* $c : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$ *then* $\Theta'(c) : (\Gamma \vdash_{\cap\cup} \Delta)$.

Proof. We prove 1, 2 and 3 by simultaneous induction on the lengths of the derivations. We consider here some of the cases in 2.

- $$\frac{}{\Gamma \mid \alpha : U_i \vdash \alpha : U_1 \cup \cdots \cup U_n, \Delta}\ (\mathsf{Ax})$$
  Let $\Gamma \vdash_{\cap\cup} M : U_i \mid \alpha : U_1 \cup \cdots \cup U_n, \Delta$. Then by the rules $(\mathsf{R}\cup)$ and (Nam), we have $[\alpha]M : (\Gamma \vdash_{\cap\cup} \alpha : U_1 \cup \cdots \cup U_n, \Delta)$ where $[\alpha]M \equiv \Theta''(M, \alpha)$.
- $$\frac{\Gamma \vdash t : I \mid \Delta \qquad \Gamma \mid e : U \vdash \Delta}{\Gamma \mid t \cdot e : I \to U \vdash \Delta}\ (\mathsf{L}\to)$$
  Let $\Gamma \vdash_{\cap\cup} M : I \to U \mid \Delta$. By the induction hypothesis, we have $\Gamma \vdash_{\cap\cup} \Theta(t) : I \mid \Delta$. Then again by the induction hypothesis, we have $\Theta''(M\Theta(t), e) : (\Gamma \vdash_{\cap\cup} \Delta)$ where $\Theta''(M\Theta(t), e) \equiv \Theta''(M, t \cdot e)$.
- $$\frac{\Gamma \mid e : U \vdash \Delta \qquad \Gamma \mid e : V \vdash \Delta}{\Gamma \mid e : U \cup V \vdash \Delta}\ (\mathsf{L}\cup)$$
  First we show the case where $e \equiv \alpha$. Then by Lemma 9, there exists $\alpha : W \in \Delta$ such that $W \equiv U \cup V \cup W_1 \cup \cdots \cup W_n$ $(n \geq 0)$. Let $\Gamma \vdash_{\cap\cup} M : U \cup V \mid \Delta$. Then by the rules $(\mathsf{R}\cup)$ and (Nam), we have $[\alpha]M : (\Gamma \vdash_{\cap\cup} \Delta)$ where $[\alpha]M \equiv \Theta''(M, \alpha)$.
  Next we show the case where $e \equiv t \cdot e'$. Then by Lemmas 7(4), $U \cup V \equiv (I_1 \to U_1) \cup \cdots \cup (I_n \to U_n)$, and by Lemma 10, for all $i \in \{1, \ldots, n\}$, $\Gamma \vdash^{-}_{\cap\cup} t : I_i \mid \Delta$ and $\Gamma \mid e' : U_1 \cup \cdots \cup U_n \vdash^{-}_{\cap\cup} \Delta$ with their derivation lengths less than that of $\Gamma \mid e : U \cup V \vdash \Delta$. Hence by the induction hypothesis, we have $\Gamma \vdash_{\cap\cup} \Theta(t) : I_i \mid \Delta$ for each $i \in \{1, \ldots, n\}$. Now let $\Gamma \vdash_{\cap\cup} M : U \cup V \mid \Delta$, and consider the following derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x : I_i \to U_i \vdash x : I_i \to U_i \mid \Delta}\ (\mathsf{Ax}) \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Theta(t) : I_i \mid \Delta\end{array}}{\Gamma, x : I_i \to U_i \vdash \Theta(t) : I_i \mid \Delta}\ \text{Lemma 1}}{\Gamma, x : I_i \to U_i \vdash x\Theta(t) : U_i \mid \Delta}\ (\to \mathsf{E})}{\Gamma, x : I_i \to U_i \vdash x\Theta(t) : U_1 \cup \cdots \cup U_n \mid \Delta}\ (\cup \mathsf{I})$$

  where x is a fresh λ-variable. Then by applying the rule $(\cup \mathsf{E})^n$ in Lemma 2, we have $\Gamma \vdash_{\cap\cup} M\Theta(t) : U_1 \cup \cdots \cup U_n \mid \Delta$. Hence by the induction hypothesis for the derivation of $\Gamma \mid e' : U_1 \cup \cdots \cup U_n \vdash \Delta$, we obtain $\Theta''(M\Theta(t), e') : (\Gamma \vdash_{\cap\cup} \Delta)$ where $\Theta''(M\Theta(t), e') \equiv \Theta''(M, t \cdot e')$. □

Next we show that reduction in the $\overline{\lambda}\mu$-calculus is simulated in the $\lambda\mu$-calculus through the translation Θ. This is used to prove one direction of the characterisation theorem of strong normalisation.

Lemma 11.

1. $\Theta(t[x := s]) \equiv \Theta(t)[x := \Theta(s)]$.
2. $\Theta'(c[x := s]) \equiv \Theta'(c)[x := \Theta(s)]$.
3. $\Theta''(M[x := \Theta(s)], e[x := s]) \equiv \Theta''(M, e)[x := \Theta(s)]$.

Proof. By simultaneous induction on the structure of t, e or c. □

In the following we abbreviate $M[\alpha \Leftarrow N_1] \cdots [\alpha \Leftarrow N_k]$ as $M[\alpha \Leftarrow N_1, \cdots, N_k]$.

Lemma 12. *Let* $e \equiv s_1 \cdot \cdots \cdot s_n \cdot \alpha$ *and* $\bar{N} \equiv \Theta(s_1), \cdots, \Theta(s_n)$.

1. $\Theta(t[\alpha := e]) \equiv \Theta(t)[\alpha \Leftarrow \bar{N}]$.
2. $\Theta'(c[\alpha := e]) \equiv \Theta'(c)[\alpha \Leftarrow \bar{N}]$.
3. $\Theta''(M[\alpha \Leftarrow \bar{N}], e'[\alpha := e]) \equiv \Theta''(M, e')[\alpha \Leftarrow \bar{N}]$.

Proof. By simultaneous induction on the structure of t, e or c. □

Lemma 13.

1. $\Theta(t[\alpha := \gamma]) \equiv \Theta(t)[\alpha := \gamma]$.
2. $\Theta'(c[\alpha := \gamma]) \equiv \Theta'(c)[\alpha := \gamma]$.
3. $\Theta''(M[\alpha := \gamma], e[\alpha := \gamma]) \equiv \Theta''(M, e)[\alpha := \gamma]$.

Proof. By simultaneous induction on the structure of t, e or c. □

Now we are in a position to show the simulation theorem.

Theorem 6.

1. *If* $t \longrightarrow_{\overline{\beta},\overline{\mu}} t'$ *then* $\Theta(t) \longrightarrow^{+}_{\beta,\mu,\rho} \Theta(t')$.
2. *If* $c \longrightarrow_{\overline{\beta},\overline{\mu}} c'$ *then* $\Theta'(c) \longrightarrow^{+}_{\beta,\mu,\rho} \Theta'(c')$.
3. *If* $e \longrightarrow_{\overline{\beta},\overline{\mu}} e'$ *then* $\Theta''(M, e) \longrightarrow^{+}_{\beta,\mu,\rho} \Theta''(M, e')$.

Proof. By simultaneous induction on the structure of t, e or c. We prove the case where the reduction is at the root.

- $\langle \lambda x.t \,|\, s \cdot e \rangle \to \langle t[x := s] \,|\, e \rangle$.
  We have $\Theta'(\langle \lambda x.t \,|\, s \cdot e \rangle) \equiv \Theta''(\Theta(\lambda x.t), s \cdot e) \equiv \Theta''((\lambda x.\Theta(t))\Theta(s), e) \longrightarrow_{\beta} \Theta''(\Theta(t)[x := \Theta(s)], e)$ and $\Theta'(\langle t[x := s] \,|\, e \rangle) \equiv \Theta''(\Theta(t[x := s]), e)$. Therefore, by Lemma 11, we have $\Theta'(\langle \lambda x.t \,|\, s \cdot e \rangle) \longrightarrow^{+}_{\beta,\mu,\rho} \Theta'(\langle t[x := s] \,|\, e \rangle)$.
- $\langle \mu\alpha.c \,|\, e \rangle \to c[\alpha := e]$.
  We have $\Theta'(\langle \mu\alpha.c \,|\, e \rangle) \equiv \Theta''(\Theta(\mu\alpha.c), e) \equiv \Theta''(\mu\alpha.\Theta'(c), e)$. Let $e \equiv s_1 \cdot \cdots \cdot s_n \cdot \gamma$ and $\bar{N} \equiv \Theta(s_1), \cdots, \Theta(s_n)$. Then, we have $\Theta''(\mu\alpha.\Theta'(c), e) \equiv \Theta''((\mu\alpha.\Theta'(c))\Theta(s_1) \cdots \Theta(s_n), \gamma) \equiv [\gamma]((\mu\alpha.\Theta'(c))\Theta(s_1) \cdots \Theta(s_n)) \longrightarrow^{*}_{\mu} [\gamma](\mu\alpha.\Theta'(c)[\alpha \Leftarrow \bar{N}]) \longrightarrow_{\rho} \Theta'(c)[\alpha \Leftarrow \bar{N}][\alpha := \gamma]$. Let $e' \equiv s_1 \cdot \cdots \cdot s_n \cdot \alpha$. Then, by Lemma 12, we have $\Theta'(c[\alpha := e']) \equiv \Theta'(c)[\alpha \Leftarrow \bar{N}]$, and so, by Lemma 13, we have $\Theta'(c[\alpha := e'][\alpha := \gamma]) \equiv \Theta'(c[\alpha := e'])[\alpha := \gamma] \equiv \Theta'(c)[\alpha \Leftarrow \bar{N}][\alpha := \gamma]$. Since $\Theta'(c[\alpha := e]) \equiv \Theta'(c[\alpha := e'][\alpha := \gamma])$, we have $\Theta'(\langle \mu\alpha.c \,|\, e \rangle) \longrightarrow^{+}_{\beta,\mu,\rho} \Theta'(c[\alpha := e])$. □

Now we can prove that all terms typable in $\overline{\lambda}\mu_{\cap\cup}$ are strongly normalising.

Corollary 3.

1. *If* $\Gamma \vdash^{-}_{\cap\cup} t : I \mid \Delta$ *for some* Γ, I *and* Δ*, then* $t \in \mathsf{SN}^{\overline{\beta},\overline{\mu}}$.
2. *If* $\Gamma \mid e : U \vdash^{-}_{\cap\cup} \Delta$ *for some* Γ, U *and* Δ*, then* $e \in \mathsf{SN}^{\overline{\beta},\overline{\mu}}$.
3. *If* $c : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$ *for some* Γ *and* Δ*, then* $c \in \mathsf{SN}^{\overline{\beta},\overline{\mu}}$.

Proof. By Theorems 5 and 6, Corollary 2, and the fact that $\mathsf{SN}^{\beta,\mu} = \mathsf{SN}^{\beta,\mu,\rho}$. The last equality follows from (i) for any term M of the $\lambda\mu$-calculus, $M \in \mathsf{SN}^{\rho}$ and (ii) if $M \longrightarrow_{\rho} M' \longrightarrow_{\beta,\mu} N$ then there exists M'' such that $M \longrightarrow_{\beta,\mu} M'' \longrightarrow^{*}_{\rho} N$. □

3.4 Characterisation of strongly normalising terms

By Corollary 3, we see that any term typable in $\overline{\lambda}\mu_{\cap\cup}$ is strongly normalising. However, it is not yet clear how many terms are typable in $\overline{\lambda}\mu_{\cap\cup}$. In this subsection we show that $\overline{\lambda}\mu_{\cap\cup}$ is powerful enough to type all strongly normalising terms. Although this is a typical property in intersection type systems for the usual λ-calculus, it has not been proved for the $\overline{\lambda}\mu$-calculus (or a fragment of the $\overline{\lambda}\mu\tilde{\mu}$-calculus) in a type system that enjoys the subject reduction property.

First we introduce notations on type environments.

Definition 8. *1. The type environment* $\Gamma \cap \Gamma'$ *is defined by*

$$\begin{aligned}\Gamma \cap \Gamma' := \{x : I \cap I' \mid x : I \in \Gamma \text{ and } x : I' \in \Gamma'\}\\ \cup \{x : I \mid x : I \in \Gamma \text{ and } x \text{ does not appear in } \Gamma'\}\\ \cup \{x : I' \mid x : I' \in \Gamma' \text{ and } x \text{ does not appear in } \Gamma\}.\end{aligned}$$

2. The type environment $\Delta \cup \Delta'$ *is defined by*

$$\begin{aligned}\Delta \cup \Delta' := \{x : U \cup U' \mid x : U \in \Delta \text{ and } x : U' \in \Delta'\}\\ \cup \{x : U \mid x : U \in \Delta \text{ and } x \text{ does not appear in } \Delta'\}\\ \cup \{x : U' \mid x : U' \in \Delta' \text{ and } x \text{ does not appear in } \Delta\}.\end{aligned}$$

Lemma 14.

1. *If* $\Gamma \vdash^{-}_{\cap\cup} t : I \mid \Delta$ *then* $\Gamma \cap \Gamma' \vdash^{-}_{\cap\cup} t : I \mid \Delta \cup \Delta'$.
2. *If* $\Gamma \mid e : I \vdash^{-}_{\cap\cup} \Delta$ *then* $\Gamma \cap \Gamma' \mid e : I \vdash^{-}_{\cap\cup} \Delta \cup \Delta'$.
3. *If* $c : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$ *then* $c : (\Gamma \cap \Gamma' \vdash^{-}_{\cap\cup} \Delta \cup \Delta')$.

Proof. By simultaneous induction on the derivations. □

Next we prove crucial lemmas about type-checking in the system $\overline{\lambda}\mu_{\cap\cup}$.

Lemma 15 (Term Inverse Substitution Lemma). *Let* $\Gamma \vdash^{-}_{\cap\cup} s : I \mid \Delta$.

1. *If* $\Gamma \vdash^{-}_{\cap\cup} t[x := s] : J \mid \Delta$ *then there exists* I' *such that* $\Gamma, x : I' \vdash^{-}_{\cap\cup} t : J \mid \Delta$ *and* $\Gamma \vdash^{-}_{\cap\cup} s : I' \mid \Delta$.

2. *If* $\Gamma \mid e[x := s] : J \vdash^{-}_{\cap\cup} \Delta$ *then there exists* I' *such that* $\Gamma, x : I' \mid e : J \vdash^{-}_{\cap\cup} \Delta$ *and* $\Gamma \vdash^{-}_{\cap\cup} s : I' \mid \Delta$.
3. *If* $c[x := s] : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$ *then there exists* I' *such that* $c : (\Gamma, x : I' \vdash^{-}_{\cap\cup} \Delta)$ *and* $\Gamma \vdash^{-}_{\cap\cup} s : I' \mid \Delta$.

Proof. By simultaneous induction on the structure of t, e or c. The non-trivial cases are the following two cases.

- $e \equiv t \cdot e'$
  Suppose $\Gamma \mid (t \cdot e')[x := s] : J \vdash^{-}_{\cap\cup} \Delta$. By Lemma 7(1), there exist $U_1 \ldots, U_n$ such that $J \equiv U_1 \cap \cdots \cap U_n$ and $\Gamma \mid (t \cdot e')[x := s] : U_i \vdash^{-}_{\cap\cup} \Delta$ for some $i \in \{1, \ldots, n\}$. Then, fixing one such i and applying Lemma 7(4), there exist $I_1, V_1, \ldots, I_m, V_m$ such that $U_i \equiv (I_1 \to V_1) \cup \cdots \cup (I_m \to V_m)$, and $\Gamma \vdash^{-}_{\cap\cup} t[x := s] : I_j \mid \Delta$ and $\Gamma \mid e'[x := s] : V_j \vdash^{-}_{\cap\cup} \Delta$ for any $j \in \{1, \ldots, m\}$. So, for each $j \in \{1, \ldots, m\}$, there exists I'_j such that $\Gamma, x : I'_j \vdash^{-}_{\cap\cup} t : I_j \mid \Delta$ and $\Gamma \vdash^{-}_{\cap\cup} s : I'_j \mid \Delta$, and there exists I''_j such that $\Gamma, x : I''_j \mid e' : V_j \vdash^{-}_{\cap\cup} \Delta$ and $\Gamma \vdash^{-}_{\cap\cup} s : I''_j \mid \Delta$, by the induction hypothesis. By Lemma 14 and ($\mathsf{L}\to$), we have $\Gamma, x : I'_j \cap I''_j \mid t \cdot e' : I_j \to V_j \vdash^{-}_{\cap\cup} \Delta$. Let $I' \equiv (I'_1 \cap I''_1) \cap \cdots \cap (I'_m \cap I''_m)$. By Lemma 14 and ($\mathsf{L}\cup$), we have $\Gamma, x : I' \mid t \cdot e' : U_i \vdash^{-}_{\cap\cup} \Delta$, and, by ($\mathsf{L}\cap$), we have $\Gamma, x : I' \mid t \cdot e' : J \vdash^{-}_{\cap\cup} \Delta$. We also have $\Gamma \vdash^{-}_{\cap\cup} s : I' \mid \Delta$ by ($\mathsf{R}\cap$).
- $c \equiv \langle t \,|\, e' \rangle$
  This case can be proved dually to the same case of the proof of Lemma 16.

□

Lemma 16 (Context Inverse Substitution Lemma). *Let* $\Gamma \mid e : U \vdash^{-}_{\cap\cup} \Delta$.

1. *If* $\Gamma \vdash^{-}_{\cap\cup} t[\alpha := e] : I \mid \Delta$ *then there exists* U' *such that* $\Gamma \vdash^{-}_{\cap\cup} t : I \mid \alpha : U', \Delta$ *and* $\Gamma \mid e : U' \vdash^{-}_{\cap\cup} \Delta$.
2. *If* $\Gamma \mid e'[\alpha := e] : I \vdash^{-}_{\cap\cup} \Delta$ *then there exists* U' *such that* $\Gamma \mid e' : I \vdash^{-}_{\cap\cup} \alpha : U', \Delta$ *and* $\Gamma \mid e : U' \vdash^{-}_{\cap\cup} \Delta$.
3. *If* $c[\alpha := e] : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$ *then there exists* U' *such that* $c : (\Gamma \vdash^{-}_{\cap\cup} \alpha : U', \Delta)$ *and* $\Gamma \mid e : U' \vdash^{-}_{\cap\cup} \Delta$.

Proof. By simultaneous induction on the structure of t, e or c. The non-trivial cases are the following two cases.

- $e \equiv t \cdot e'$
  This case can be proved dually to the same case of the proof of Lemma 15.
- $c \equiv \langle t \,|\, e' \rangle$
  Suppose $\langle t \,|\, e' \rangle[\alpha := e] : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$. Then, there exists I such that $\Gamma \vdash^{-}_{\cap\cup} t[\alpha := e] : I \mid \Delta$ and $\Gamma \mid e'[\alpha := e] : I \vdash^{-}_{\cap\cup} \Delta$. By the induction hypothesis, there exists V' such that $\Gamma \vdash^{-}_{\cap\cup} t : I \mid \alpha : V', \Delta$ and $\Gamma \mid e : V' \vdash^{-}_{\cap\cup} \Delta$, and there exists V'' such that $\Gamma \mid e' : I \vdash^{-}_{\cap\cup} \alpha : V'', \Delta$ and $\Gamma \mid e : V'' \vdash^{-}_{\cap\cup} \Delta$. Let $U' \equiv V' \cup V''$. By Lemma 14 and (Cut), we have $\langle t \,|\, e' \rangle : (\Gamma \vdash^{-}_{\cap\cup} \alpha : U', \Delta)$, and by ($\mathsf{L}\cup$), we have $\Gamma \mid e : U' \vdash^{-}_{\cap\cup} \Delta$. □

Now we prove the characterisation theorem of strongly normalising terms.

Theorem 7.

1. $t \in \mathsf{SN}^{\overline{\beta},\overline{\mu}}$ *if and only if* $\Gamma \vdash^{-}_{\cap\cup} t : I \mid \Delta$ *for some* Γ, I *and* Δ.
2. $e \in \mathsf{SN}^{\overline{\beta},\overline{\mu}}$ *if and only if* $\Gamma \mid e : U \vdash^{-}_{\cap\cup} \Delta$ *for some* Γ, U *and* Δ.
3. $c \in \mathsf{SN}^{\overline{\beta},\overline{\mu}}$ *if and only if* $c : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$ *for some* Γ *and* Δ.

Proof. The right to left implications are by Corollary 3. For the converses, we prove 1, 2 and 3 simultaneously by main induction on the maximal length of all $\overline{\beta}, \overline{\mu}$-reduction sequences out of t, e or c, and subinduction on the structure of t, e or c. We analyse the possible cases according to the shape of t, e or c.

- $t \equiv x$ for some λ-variable x. In this case we just have to take $x : I \vdash x : I \mid$ which is an axiom.
- $e \equiv \alpha$ for some μ-variable α. Similar, taking an axiom $\mid \alpha : U \vdash \alpha : U$.
- $t \equiv \lambda x.s$. By the subinduction hypothesis, there exist Γ, I and Δ such that $\Gamma \vdash^{-}_{\cap\cup} s : I \mid \Delta$. Let $I \equiv U_1 \cap \cdots \cap U_n$. Then by Lemma 4(1), we have $\Gamma \vdash^{-}_{\cap\cup} s : U_i \mid \Delta$ for any $i \in \{1, \ldots, n\}$. Hence, if $x : J \in \Gamma$ then we have $\Gamma \setminus \{x : J\} \vdash^{-}_{\cap\cup} \lambda x.s : J \to U_i \mid \Delta$. Otherwise, using Lemma 14(1), we have $\Gamma \vdash^{-}_{\cap\cup} \lambda x.s : J \to U_i \mid \Delta$ for some J.
- The cases $t \equiv \mu\alpha.c$, $e \equiv t \cdot e'$, $c \equiv \langle x \,|\, e \rangle$ and $c \equiv \langle \lambda x.s \,|\, \alpha \rangle$ are proved using the subinduction hypothesis and Lemma 14.

The last two cases require us to use the main induction hypothesis.

- $c \equiv \langle \mu\alpha.c' \,|\, e \rangle$. By the main induction hypothesis, there exist Γ and Δ such that $c'[\alpha := e] : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$. By the subinduction hypothesis, there exist Γ', U and Δ' such that $\Gamma' \mid e : U \vdash^{-}_{\cap\cup} \Delta'$. Then by Lemmas 14 and 16(3), there exists U' such that $c' : (\Gamma \cap \Gamma' \vdash^{-}_{\cap\cup} \alpha : U', \Delta \cup \Delta')$ and $\Gamma \cap \Gamma' \mid e : U' \vdash^{-}_{\cap\cup} \Delta \cup \Delta'$. From the former, we have $\Gamma \cap \Gamma' \vdash^{-}_{\cap\cup} \mu\alpha.c' : U' \mid \Delta \cup \Delta'$. Hence by the rule (Cut), we obtain $\langle \mu\alpha.c' \,|\, e \rangle : (\Gamma \cap \Gamma' \vdash^{-}_{\cap\cup} \Delta \cup \Delta')$.
- $c \equiv \langle \lambda x.t \,|\, s \cdot e \rangle$. By the main induction hypothesis, there exist Γ and Δ such that $\langle t[x := s] \,|\, e \rangle : (\Gamma \vdash^{-}_{\cap\cup} \Delta)$. Then there exists J such that $\Gamma \vdash^{-}_{\cap\cup} t[x := s] : J \mid \Delta$ and $\Gamma \mid e : J \vdash^{-}_{\cap\cup} \Delta$. By the subinduction hypothesis, there exist Γ', I and Δ' such that $\Gamma' \vdash^{-}_{\cap\cup} s : I \mid \Delta'$. Hence by Lemmas 14 and 15(1), there exists I' such that $\Gamma \cap \Gamma', x : I' \vdash^{-}_{\cap\cup} t : J \mid \Delta \cup \Delta'$ and $\Gamma \cap \Gamma' \vdash^{-}_{\cap\cup} s : I' \mid \Delta \cup \Delta'$. From the former, we have $\Gamma \cap \Gamma' \vdash^{-}_{\cap\cup} \lambda x.t : I' \to J \mid \Delta \cup \Delta'$. From the latter and $\Gamma \mid e : J \vdash^{-}_{\cap\cup} \Delta$, we have $\Gamma \cap \Gamma' \mid s \cdot e : I' \to J \vdash^{-}_{\cap\cup} \Delta \cup \Delta'$. Hence by the rule (Cut), we obtain $\langle \lambda x.t \,|\, s \cdot e \rangle : (\Gamma \cap \Gamma' \vdash^{-}_{\cap\cup} \Delta \cup \Delta')$. □

4 Conclusion

We have presented a translation from intersection and union types into intersection and product types. Using the translation, we have shown that our intersection and union type system for the $\lambda\mu$-calculus can be embedded into the type system of [vBBdL13], which yields strong normalisation of terms typable by our system. We have also presented an intersection and union type system for the

$\overline{\lambda}\mu$-calculus, and proved the subject reduction property and the characterisation theorem of strong normalisation.

It is expected that our type system for the $\lambda\mu$-calculus enjoys the subject reduction property. This is plausible since, by the side condition of the union-elimination rule, the variable to be discharged can occur only once in each premiss, in which case known counter examples do not emerge. It is also expected that all strongly normalising terms in the $\lambda\mu$-calculus are typable in our system. These problems are to be investigated in future work.

Another direction for future work is to design intersection and union type systems for call-by-value languages based on duality in sequent calculus. For languages without control operators, some natural deduction style systems have been proposed [DP03,Rib09]. To give a uniform perspective, however, we consider the sequent calculus approach to be promising.

Acknowledgements. We would like to thank the anonymous referees for detailed comments and useful suggestions. The figures of the derivations have been drawn using Makoto Tatsuta's `proof.sty` macros. The second author was supported by JSPS KAKENHI Grant Numbers 24650002 and 25280025.

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A Preservation of derivability in $\boldsymbol{\lambda\mu}_{\cap\cup}$

In this section we prove the preservation of derivability in $\lambda\mu_{\cap\cup}$ by the translation in Subsection 2.4.

To simplify the argument, we first restrict the relations $\leq_D$ and $\leq_C$ to $\leq_D^{\#}$ and $\leq_C^{\#}$ defined by the rules in Figure 6 (where $\leq_A^{\#}$ denotes either $\leq_D^{\#}$ or $\leq_C^{\#}$). Clearly, $\sigma \leq_A^{\#} \tau$ implies $\sigma \leq_A \tau$.

$$\frac{}{\sigma \leq_A^{\#} \sigma} \qquad \frac{}{\sigma \wedge \tau \leq_A^{\#} \sigma} \qquad \frac{}{\sigma \wedge \tau \leq_A^{\#} \tau}$$

$$\frac{\sigma \leq_A^{\#} \rho \quad \rho \leq_A^{\#} \tau}{\sigma \leq_A^{\#} \tau} \qquad \frac{\sigma \leq_A^{\#} \tau_1 \quad \sigma \leq_A^{\#} \tau_2}{\sigma \leq_A^{\#} \tau_1 \wedge \tau_2}$$

$$\frac{\delta_1 \leq_D^{\#} \delta_2 \quad \kappa_1 \leq_C^{\#} \kappa_2}{\delta_1 \times \kappa_1 \leq_C^{\#} \delta_2 \times \kappa_2} \qquad \frac{\kappa_2 \leq_C^{\#} \kappa_1}{\kappa_1 \to \nu \leq_D^{\#} \kappa_2 \to \nu}$$

Fig. 6. Relations $\leq_D^{\#}$ and $\leq_C^{\#}$

Lemma 17. *Let $\delta_1 \wedge \cdots \wedge \delta_n \leq_D^{\#} \delta'_1 \wedge \cdots \wedge \delta'_m$ where none of the δ_i ($i \in \{1, \ldots, n\}$) and δ'_j ($j \in \{1, \ldots, m\}$) is an intersection. Then, for each δ'_j of the form $\kappa' \to \nu$, there exists δ_i such that $\delta_i \equiv \kappa \to \nu$ and $\kappa' \leq_C^{\#} \kappa$.*

Proof. By induction on the definition of $\leq_D^{\#}$. □

The next lemma is the $\leq^{\#}$ analogue of Lemma 3.

Lemma 18. $(\delta_1 \wedge \cdots \wedge \delta_n) \times (\kappa_1 \wedge \cdots \wedge \kappa_n) \leq_C^{\#} (\delta_1 \times \kappa_1) \wedge \cdots \wedge (\delta_n \times \kappa_n)$.

The type assignment system $\lambda\mu_{\wedge\times}^{\#}$ is defined by the same rules as $\lambda\mu_{\wedge\times}$ except that the rule $(\leq)$ is replaced by the rule $(\leq^{\#})$ with $\leq_D^{\#}$ instead of $\leq_D$. (We also assume $\kappa \not\equiv \omega$ in the rules (Abs) and (App).) We write $\Gamma \vdash_{\wedge\times}^{\#} M : \delta \mid \Delta$ if $\Gamma \vdash M : \delta \mid \Delta$ is derivable in the system $\lambda\mu_{\wedge\times}^{\#}$. Clearly, $\Gamma \vdash_{\wedge\times}^{\#} M : \delta \mid \Delta$ implies $\Gamma \vdash_{\wedge\times} M : \delta \mid \Delta$.

Lemma 19. *If $\Gamma, x : \delta \vdash_{\wedge\times}^{\#} M : \delta'$ and $x \notin \mathsf{FV}_\lambda(M)$ then $\Gamma \vdash_{\wedge\times}^{\#} M : \delta'$.*

Proof. By induction on the derivation of $\Gamma, x : \delta \vdash_{\wedge\times}^{\#} M : \delta'$. □

The next lemma is straightforward, but important for proving Theorem 8.

Lemma 20 (Generation Lemma).

1. *If $\Gamma \vdash_{\wedge\times}^{\#} x : \delta$ then there exists $x : \delta' \in \Gamma$ such that $\delta' \leq_D^{\#} \delta$.*
2. *If $\Gamma \vdash_{\wedge\times}^{\#} MN : \delta \mid \Delta$ then there exist $\kappa_1, \ldots, \kappa_n, \delta_1, \ldots, \delta_n$ ($n \geq 1$) such that $(\kappa_1 \to \nu) \wedge \cdots \wedge (\kappa_n \to \nu) \leq_D^{\#} \delta$ and, for all $i \in \{1, \ldots, n\}$, $\Gamma \vdash_{\wedge\times}^{\#} M : \delta_i \times \kappa_i \to \nu \mid \Delta$ and $\Gamma \vdash_{\wedge\times}^{\#} N : \delta_i \mid \Delta$.*

Proof. By induction on the derivations. □

Now we prove the announced theorem.

Theorem 8. *If* $\Gamma \vdash_{\cap\cup} M : I \mid \Delta$ *then* $\Gamma^D \vdash^{\#}_{\wedge\times} M : I^D \mid \Delta^C$.

Proof. By induction on the derivation of $\Gamma \vdash_{\cap\cup} M : I \mid \Delta$. Here we only consider the case where the last applied rule in the derivation is $(\cup\mathsf{E})$. The other cases are proved similarly to those in the proof of Theorem 2 by checking that the derivations under consideration only use the rules of $\lambda\mu^{\#}_{\wedge\times}$.

- $$\frac{\Gamma \vdash M : U \cup V \mid \Delta \quad \Gamma, x : U \vdash xN : I \mid \Delta \quad \Gamma, x : V \vdash xN : I \mid \Delta}{\Gamma \vdash MN : I \mid \Delta}\ (\cup\mathsf{E})$$

  where $x \notin \mathsf{FV}_\lambda(N)$. By IH, we have $\Gamma^D \vdash^{\#}_{\wedge\times} M : (U \cup V)^D \mid \Delta^C$, $\Gamma^D, x : U^D \vdash^{\#}_{\wedge\times} xN : I^D \mid \Delta^C$ and $\Gamma^D, x : V^D \vdash^{\#}_{\wedge\times} xN : I^D \mid \Delta^C$. Then by Lemma 20(2), we have

  (a) there exist $\kappa_1, \ldots, \kappa_n, \delta_1, \ldots, \delta_n$ $(n \geq 1)$ such that $(\kappa_1 \to \nu) \wedge \cdots \wedge (\kappa_n \to \nu) \leq^{\#}_D I^D$ and, for all $i \in \{1, \ldots, n\}$, $\Gamma^D, x : U^D \vdash^{\#}_{\wedge\times} x : \delta_i \times \kappa_i \to \nu \mid \Delta^C$ and $\Gamma^D, x : U^D \vdash^{\#}_{\wedge\times} N : \delta_i \mid \Delta^C$, and

  (b) there exist $\kappa'_1, \ldots, \kappa'_{n'}, \delta'_1, \ldots, \delta'_{n'}$ $(n' \geq 1)$ such that $(\kappa'_1 \to \nu) \wedge \cdots \wedge (\kappa'_{n'} \to \nu) \leq^{\#}_D I^D$ and, for all $k \in \{1, \ldots, n'\}$, $\Gamma^D, x : V^D \vdash^{\#}_{\wedge\times} x : \delta'_k \times \kappa'_k \to \nu \mid \Delta^C$ and $\Gamma^D, x : V^D \vdash^{\#}_{\wedge\times} N : \delta'_k \mid \Delta^C$.

  Let $I^D \equiv (U_1 \cap \cdots \cap U_m)^D \equiv U_1^D \wedge \cdots \wedge U_m^D \equiv (U_1^C \to \nu) \wedge \cdots \wedge (U_m^C \to \nu)$. Then, from (a) and Lemma 17, we have

  (c) for each $U_j^D (\equiv U_j^C \to \nu)$ $(j \in \{1, \ldots, m\})$, there exists κ_i $(i \in \{1, \ldots, n\})$ such that
    (c1) $U_j^C \leq^{\#}_C \kappa_i$,
    (c2) $\Gamma^D, x : U^D \vdash^{\#}_{\wedge\times} x : \delta_i \times \kappa_i \to \nu \mid \Delta^C$, and
    (c3) $\Gamma^D, x : U^D \vdash^{\#}_{\wedge\times} N : \delta_i \mid \Delta^C$.

  Similarly, from (b) and Lemma 17, we have

  (d) for each $U_j^D (\equiv U_j^C \to \nu)$ $(j \in \{1, \ldots, m\})$, there exists κ'_k $(k \in \{1, \ldots, n'\})$ such that
    (d1) $U_j^C \leq^{\#}_C \kappa'_k$,
    (d2) $\Gamma^D, x : V^D \vdash^{\#}_{\wedge\times} x : \delta'_k \times \kappa'_k \to \nu \mid \Delta^C$, and
    (d3) $\Gamma^D, x : V^D \vdash^{\#}_{\wedge\times} N : \delta'_k \mid \Delta^C$.

  Now, fix $j \in \{1, \ldots, m\}$, and take κ_i and κ'_k satisfying (c1)-(c3) and (d1)-(d3), respectively. From (c2), we have $U^D \leq^{\#}_D \delta_i \times \kappa_i \to \nu$ by Lemma 20(1). Since $U^D \equiv U^C \to \nu$, we have $\delta_i \times \kappa_i \leq^{\#}_C U^C$ by Lemma 17. Similarly, from (d2), we have $\delta'_k \times \kappa'_k \leq^{\#}_C V^C$. Hence $(\delta_i \times \kappa_i) \wedge (\delta'_k \times \kappa'_k) \leq^{\#}_C U^C \wedge V^C$, and so

  $$\begin{aligned} U^C \wedge V^C \to \nu &\leq^{\#}_D (\delta_i \times \kappa_i) \wedge (\delta'_k \times \kappa'_k) \to \nu \\ &\leq^{\#}_D (\delta_i \wedge \delta'_k) \times (\kappa_i \wedge \kappa'_k) \to \nu \qquad \text{(by Lemma 18)} \end{aligned}$$

By IH, we have $\varGamma^D \vdash^{\#}_{\wedge\times} M : (U \cup V)^D \mid \varDelta^C$ where $(U \cup V)^D \equiv (U \cup V)^C \to \nu \equiv U^C \wedge V^C \to \nu$. So we have $\varGamma^D \vdash^{\#}_{\wedge\times} M : (\delta_i \wedge \delta'_k) \times (\kappa_i \wedge \kappa'_k) \to \nu \mid \varDelta^C$ by the rule $(\leq^{\#})$. On the other hand, from (c3) and (d3), we have $\varGamma^D \vdash^{\#}_{\wedge\times} N : \delta_i \mid \varDelta^C$ and $\varGamma^D \vdash^{\#}_{\wedge\times} N : \delta'_k \mid \varDelta^C$ by Lemma 19. Thus we have the following derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \varGamma^D \vdash M : (\delta_i \wedge \delta'_k) \times (\kappa_i \wedge \kappa'_k) \to \nu \mid \varDelta^C\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \varGamma^D \vdash N : \delta_i \mid \varDelta^C\end{array} \quad \begin{array}{c}\vdots\\ \varGamma^D \vdash N : \delta'_k \mid \varDelta^C\end{array}}{\varGamma^D \vdash N : \delta_i \wedge \delta'_k \mid \varDelta^C}\,(\wedge)}{\varGamma^D \vdash MN : \kappa_i \wedge \kappa'_k \to \nu \mid \varDelta^C}\,(\mathsf{App})$$

Finally, from (c1) and (d1), we have $U_j^C \leq^{\#}_C \kappa_i \wedge \kappa'_k$, and so $\kappa_i \wedge \kappa'_k \to \nu \leq^{\#}_D U_j^C \to \nu \equiv U_j^D$. Hence by the rule $(\leq^{\#})$, we obtain $\varGamma^D \vdash^{\#}_{\wedge\times} MN : U_j^D \mid \varDelta^C$. This argument holds for each U_j^D $(j \in \{1, \ldots, m\})$, so we have $\varGamma^D \vdash^{\#}_{\wedge\times} MN : U_1^D \wedge \cdots \wedge U_m^D \mid \varDelta^C$. Hence $\varGamma^D \vdash^{\#}_{\wedge\times} MN : I^D \mid \varDelta^C$. □

Corollary 2. *If $\varGamma \vdash_{\cap\cup} M : I \mid \varDelta$ for some $\varGamma, I$ and $\varDelta$, then $M \in \mathsf{SN}^{\beta,\mu}$.*

Proof. By Theorems 1 and 8, and the fact that $\varGamma^D \vdash^{\#}_{\wedge\times} M : I^D \mid \varDelta^C$ implies $\varGamma^D \vdash_{\wedge\times} M : I^D \mid \varDelta^C$. □