Non-\(E\)-overlapping and weakly shallow TRSs are confluent (Extended abstract)

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1 Introduction

Confluence of term rewriting systems (TRSs) is undecidable, even for flat TRSs [MOJ06] or length-two string rewrite systems [SW08]. Two decidable subclasses are known: right-linear and shallow TRSs by tree automata techniques [GT05] and terminating TRSs [KB70]. Most of sufficient conditions are for either terminating TRSs [KB70] (extended to TRSs with relative termination [HAI1, KH12]) or left-linear non-overlapping TRSs (and their extensions) [Row73, Huc80, Toy87, Oos85, Oku98, Oo97]. For non-linear TRSs, a goal is RTA open problem [58] “strongly non-overlapping and right-linear TRSs are confluent”. A best known result strengthens the right-linear assumption to simple-right-linear [TO95], which means that each rewrite rule is right-linear and no left-non-linear variables appear in the right hand side. Other trials by depth-preserving conditions are found in [GOO98].

We have proposed a different methodology, called a reduction graph [SO10]. It has shown that “weakly non-overlapping, shallow, and non-collapsing TRSs are confluent.” An original idea comes from observation that, when non-\(E\)-overlapping, peak-elimination uses only “copies” of reductions in an original rewrite sequences. Thus, if we focus on terms appearing in peak elimination, they are finitely many. We regard a rewrite relation over these terms as a directed graph, and we construct a confluent directed acyclic graph (DAG) in a bottom-up manner, in which the shallow assumption works. The keys are, a connected convergent DAG always has a unique normal form (if it is finite), and convergence is preserved if we add an arbitrary reduction starting from that normal form.

This paper briefly sketches that “non-\(E\)-overlapping and weakly-shallow TRSs are confluent” by extending reduction graph in our previous work [SO10] by introducing constructor expansion. A term is weakly shallow if each defined function symbol appears either at the root or in the ground subterms, and a TRS is weakly shallow if the both sides of rules are weakly shallow. The non-\(E\)-overlapping property is undecidable for weakly shallow TRSs [MOM12] and a decidable sufficient condition is the strongly non-overlapping condition. A Turing machine can be simulated by a weakly shallow TRS (p.27 in [KL03]); thus the word problem is undecidable, in contrast to shallow TRSs [CHJ94].

Basic definitions and notations

We follow standard definitions and terminology of graphs and TRSs [BN98]. As notational convention, \(V\) for a finite set (often of terms), \(F\) is a finite set of function symbols, \(D\) and \(C\) are the sets of defined and constructor symbols in \(F\), respectively. \(X\) is the set of variables. We use \(s, t, u, v, w\) for terms, \(x, y\) for variables, \(p, q\) for positions, \(\sigma, \theta\) for substitutions, \(\ell \to r\) for a rewrite rule, and \(R\) for a TRS.

An abstract reduction system (ARS) is a directed graph \(G = (V, \to)\) with \(\to \subseteq V \times V\). For \(V', V'' \subseteq V\), \(\to_{V' \times V''} \equiv \to \cap (V' \times V'')\). We write \(V_{G}\) and \(\to_{G}\) to emphasize \(G\). An edge \(v \to u\) is an out-edge of \(v\) and an in-edge of \(u\). A node \(v\) is normal if it has no out-edges. Let \(G = (V, \to)\) and \(G' = (V', \to')\). The union \(G \cup G'\) is \((V \cup V', \to \cup \to')\). We say \(G\) is finite if \(V\) is finite, \(G\) is convergent if \(G\) is confluent and terminating, \(G'\) includes \(G\) (denoted by \(G' \supseteq G\)) if \(V' \supseteq V\) and \(\to' \supseteq \to\), and \(G'\) weakly subsumes \(G\) (denoted by \(G' \supseteq \ast G\)) if \(V' \supseteq V\) and \(\to' \supseteq \ast \to\).

We use \(\text{sub}(t)\) for the set of direct subterms of a term \(t\) defined as \(\text{sub}(t) = \emptyset\) if \(t\) is a variable and \(\text{sub}(t) = \{t_1, \ldots, t_n\}\) if \(t = f(t_1, \ldots, t_n)\). \(\frac{p}{R} t\) is a top reduction if \(p = \varepsilon\). Otherwise, it is a non-top
reduction, written as \( s{\xrightarrow{\text{R}}} t \). We use \( T|_f \) to denote the subset of \( T \subseteq T(F, X) \) and \( f \in F \) that consists of the terms in \( T \) with the root symbol \( f \). For \( F' \subseteq F \), we use \( T|_{F'} \) to denote \( \cup_{f \in F'} T|_f \).

A **weakly shallow term** is a term in which defined function symbols appear only either at the root or in the ground subterms (i.e., \( p \neq \varepsilon \) and \( \text{root}(s|_p) \in D \) imply that \( s|_p \) is ground). A rewrite rule \( \ell \to r \) is weakly shallow if \( \ell \) and \( r \) are weakly shallow. A TRS is weakly shallow if each rewrite rule is weakly shallow. We assume that a TRS has finitely many rewrite rules.

Let \( \ell_1 \to r_1, \ell_2 \to r_2 \in R \). If there exist substitutions \( \theta_1, \theta_2 \) such that \( \ell_1|_p \theta_1 = \ell_2|_p \theta_2 \) (resp. \( \ell_1|_p \theta_1 \xrightarrow{\varepsilon} \ell_2|_p \theta_2 \), \( (r_1|_p \theta_1, (\ell_1|_p \theta_1)|_p \theta_2|_p) \) is a critical pair (resp. E-critical pair) except that \( p = \varepsilon \) and the two rules are identical (up to renaming variables). A TRS \( R \) is overlapping (resp. E-overlapping, strongly overlapping) if there exists a critical pair (resp. E-critical pair, critical pair of linearization of \( R \)). Note that when a TRS is left-linear, they are equivalent.

## 2 Extensions of convergent abstract reduction systems

**Definition 2.1.** For ARSs \( G_1 = (V_1, \to_1) \) and \( G_2 = (V_2, \to_2) \), we say that \( G_1 \cup G_2 \) is the **hierarchical combination** of \( G_2 \) with \( G_1 \), denoted by \( G_1 \supset G_2 \), if \( \to \subseteq (V_1 \setminus V_2) \times V_2 \).

**Lemma 2.2.** Let \( G_1 \supset G_2 \) be a convergent hierarchical combination of ARSs. If a convergent ARS \( G_3 \) weakly subsumes \( G_2 \) and \( G_1 \supset G_3 \) is a hierarchical combination, then \( G_1 \supset G_3 \) is convergent.

**Definition 2.3.** Let \( G = (V, \to) \) be a convergent ARS and \( v \neq v' \). Let \( G' \) be obtained by:

\[
\begin{align*}
&\{ v \cup \{v'\}, \to \cup \{(v,v')\} \} & \text{if } v \in V & \text{is } \to\text{-normal, and } v' \notin V \\
&\{ v, \to \cup \{(v,v')\} \} & \text{if } v \in V & \text{is } \to\text{-normal, } v' \in V \text{ and } v' \not\leftrightarrow v \\
&\{ v, \to \cup \{(v',v) | v' \to v', v \notin V \} \cup \{(v,v')\} \} & \text{if } v \in V & \text{is } \to\text{-normal, } v' \in V, \text{ and } v' \not\leftrightarrow v \\
&\{ v \cup \{v',v''\}, \to \cup \{(v,v')\} \} & \text{if } v \notin V \\
&\text{Undefined} & \text{otherwise}
\end{align*}
\]

We denote \( G' \) by \( (V, \to) \to (v \to v') \) if \( G' \) is defined (i.e., the first four cases). We denote \( G \to (v_0 \to v_1) \to (v_1 \to v_2) \to \cdots \to (v_{n-1} \to v_n) \) as \( G \to (v_0 \to v_1 \to \cdots \to v_n) \).

**Proposition 2.4.** Let \( G = (V, \to) \) be a convergent ARS. Let \( v_0, v_1, \ldots, v_n \) satisfy \( v_i \neq v_j \) (for \( i \neq j \)), and the following conditions:

i) if \( v_0 \in V \), then \( v_0 \) is \( \to \)-normal and \( v_i \in V \) implies \( v_i \not\leftrightarrow v_0 \) for each \( i < n \),

ii) if \( v_0 \notin V \), then \( v_1, \ldots, v_{n-1} \notin V \).

Then, \( G' = G \to (v_0 \to v_1 \to \cdots \to v_n) \) is convergent, and satisfies \( G' \supseteq G \).

## 3 Reduction graphs

**Definition 3.1 (SOIII).** A finite ARS \( G = (V, \to) \) is an R-reduction graph if \( V \subseteq T(F, X) \) and \( \to \subseteq \to_R \).

For an R-reduction graph \( G = (V, \to) \), top-edges, inner-edges, and strict inner-edges are given as \( \xrightarrow{\text{t}} = \to \cap \xrightarrow{\varepsilon} \), \( \xrightarrow{\text{i}} = \to \cap \xrightarrow{\varepsilon_R} \), and \( \xrightarrow{\text{s}} = \to \setminus \xrightarrow{\varepsilon} \), respectively. We use \( G^t, G^i, \) and \( G^s \) to denote \( \langle V, \xrightarrow{\text{t}} \rangle, \langle V, \xrightarrow{\text{i}} \rangle, \) and \( \langle V, \xrightarrow{\text{s}} \rangle \), respectively. Remark that an edge \( (s, t) \in \to \) may be both \( \xrightarrow{\text{t}} \) and \( \xrightarrow{\varepsilon} \), e.g., \( (f(a, a), f(b, a)) \) for \( R = \{ a \to b, f(x, x) \to f(b, a) \} \). For an R-reduction graph \( G = (V, \to) \) and \( F' \subseteq F \), we represent \( G|_{F'} = (V, \to|_{F'}) \) where \( \to|_{F'} = \to|_{V \times F'} \).
Definition 3.2. Let $G = \langle V, \rightarrow \rangle$ be an $R$-reduction graph. The direct-subterm reduction-graph sub(G) of G is (sub(V), sub(→)) where (sub(V), sub(→)) = (∪_{t \in V} sub(t), \{ (s_i, t_i) \mid f(s_1, \ldots, s_n) \preceq f(t_1, \ldots, t_n), s_i \neq t_i \}). An R-reduction graph $G = \langle V, \rightarrow \rangle$ is subterm-closed if sub(V) ⊆ V and sub(\{f \} \subseteq \leftrightarrow^{*}.

Lemma 3.3. Let $G = \langle V, \rightarrow \rangle$ be a subterm-closed R-reduction graph. Assume that $p \in \text{Pos}(s)$ for a term $s$ and $s[t_p] \leftrightarrow^{*} s[t'_p]$, in which any reductions do not occur above $p$. Then $t \leftrightarrow^{*} t'$.

Definition 3.4. Let $G = \langle V, \rightarrow \rangle$ be an R-reduction graph and $F' (\subseteq F)$. The $F'$-monotonic extension is $M_{F'}(G) = \langle V_1, \rightarrow_1 \rangle$ for

\[
V_1 = \{ f(s_1, \ldots, s_n) \mid f \in F', s_1, \ldots, s_n \in V \},
\]

\[
\rightarrow_1 = \{ \langle f(\ldots s \ldots), f(\ldots t \ldots) \rangle \in V_1 \times V_1 \mid s \rightarrow t \}.
\]

When $G$ is subterm-closed, an $C$-expansion $\overline{M_C}(G)$ is the hierarchical combination $G_D > M_C(G) (= G_D \cup M_C(G))$. The k-times application of $\overline{M_C}$ to $G$ is denoted by $\overline{M_C}^k(G)$.

Example 3.5. As a running example, we use a TRS $R_2 = \{ f(x, g(x)) \rightarrow g^3(x), c \rightarrow g(c) \}$ with $C = \{ g \}$ and $D = \{ c, f \}$. Consider a subterm-closed $R_2$-reduction graph $G = \{ \langle c, g(c), g^3(c) \rangle, \langle c, g(c) \rangle \}$. For easy description, we also denote as $G = \{ c \rightarrow g(c), g^3(c) \}$. Then, $M_C(G) = \{ c \rightarrow g^3(c), g^3(c) \}$. $\overline{M_C^2}(G) = \{ c \rightarrow g(c) \rightarrow g^3(c), g^3(c) \}, \overline{M_C^3}(G) = \{ c \rightarrow g(c) \rightarrow g^3(c) \rightarrow g^3(c), g^3(c) \}$.

Lemma 3.6. For a subterm-closed R-reduction graph $G$ and $m > k \geq 0$, (1) $G \subseteq \overline{M_C}^k(G)$, (2) $\overline{M_C}^k(G)$ is subterm-closed, (3) $\overline{M_C}(G)$ is convergent, if $G$ is convergent, and (4) $\overline{M_C^0}(G) \subseteq \overline{M_C^0}(G)$.

4 Constructor expansion

In Section 3 and 5, given an R-reduction graph $G_0$, we show how to inductively construct a convergent and subterm-closed R-reduction graph $G_4$ with $G_0 \subseteq G_4$. Note that Section 5 assumes that a TRS $R$ is non-E-overlapping and weakly shallow. Throughout these sections, we fix the notations.

- Given an R-reduction graph $G_0 = \langle V_0, \rightarrow_0 \rangle$ as an input.
- $G = \langle V, \rightarrow \rangle$ is used to denote a convergent and subterm-closed R-reduction graph that weakly subsumes sub($G_0$) (by induction hypothesis).
- $G_1 = \langle V_1, \rightarrow_1 \rangle$ denotes a convergent R-reduction graph with $M_{F'}(G) \subseteq G_1$ (by Lemma 4.1).
- $G_2, = \langle V_2, \rightarrow_2 \rangle$ denotes $M_{F'}(\overline{M_C^{i-2}}(G))$ for $i \geq 0$.
- $T$ denotes a subgraph of $(G^0_0 \cup G^1_0 \cup G^2_0) \setminus (G^0_0 \cup G^1_0 \cup G^2_0)$ such that $T$ modulo $\leftrightarrow^*_1$ is acyclic and preserves connectivity of $(G^0_0 \cup G^1_0) \setminus (G^0_0 \cup G^1_0)$ modulo $\leftrightarrow^*_1$.
- We repeatedly expand $G_1$ by adding edges of $T$ from nodes with out-edges only to sink order, and construct a convergent and subterm-closed $G_4$ with $G_0 \subseteq G_4$.

If $G_1 = \langle V_1, \rightarrow_1 \rangle$ is convergent, we refer the normal form (in $G_1$) of a term $u \in V_1$ by $u_{\rightarrow_1}$.

Lemma 4.1. For a convergent and subterm-closed R-reduction graph $G$, there exist $k \geq 0$ and an R-reduction graph $G_1$ satisfying the following conditions: i) $G_1$ is convergent, and consists of non-top edges, ii) $G_1 \subseteq G_{2k}$, iii) $u \leftrightarrow_{2k} v$ imply $u \leftrightarrow_{2k} v$ for each $u, v \in V_1$ and $i \geq 0$, and iv) $M_{F'}(G) \subseteq G_1$.

Example 4.2. Consider $R_2$ in Example 3.5. Let $G_0 = \{ f(g(c), c) \rightarrow f(c, g(c)) \rightarrow g^3(c) \}$. The subterm graph sub($G_0$) is equal to $G$ in Example 3.5 and is convergent and subterm-closed. Then, Lemma 4.1 starts from $M_{F'}(G)$, which is displayed by the solid arrows in Figure 4. An example of $G_1$ is constructed by augmenting the dashed edges with $k = 1$. 

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Let us consider to apply Lemma 5.2 on Example 5.3.

Let \( G = \langle V, \rightarrow \rangle \) be an \( R \)-convergent subterm-closed \( R \)-reduction graph, \( G_1 = \langle V_1, \rightarrow_1 \rangle \), and \( k \) as in Lemma 4.1. Let \( \rightarrow_s, \rightarrow_T \subseteq V_1 \times V_1 \) such that \( \rightarrow_s = \rightarrow_s^\rightarrow, \rightarrow_T = \rightarrow_T^\rightarrow, \rightarrow_k = (\rightarrow_s \cup \leftrightarrow_T \cup \leftrightarrow_1)^k \), and

\( v) \) The component graph \( (S \cup T)/G_1 \) is acyclic, where out-edges are at most one for each node. Moreover, if \( [u]_{\rightarrow_1^+} \) has an in-edge in \( T/G_1 \) then it has no edges in \( S/G_1 \).

\( vi) \) \( u \) is \( \rightarrow_1 \)-normal for each \( (u, v) \in S \).

If \( T \neq \emptyset \), there is a tuple \((S', T', G_1', k') \) such that \( |T'| > |T| \) and the conditions of \( v) \) to \( vi) \), \( (1) \leftrightarrow_1^* \subseteq \leftrightarrow_1^*, \) and \( (2) (\rightarrow_1^* \cup \leftrightarrow_s)^k \subseteq (\rightarrow_T \cup \leftrightarrow_S \cup \leftrightarrow_1)^k \) hold. We denote it by \((S, T, G_1, k) \vdash (S', T', G_1', k') \).

A convergent reduction graph \( G_4 = \langle V_4, \rightarrow_4 \rangle \) with \( G_0 \subseteq G_4 \) is obtained from \( S = \emptyset, T \) (after \( \vdash \) in Lemma 5.2 is preprocessed), and \( G_1 \) by repeated applications of \( \vdash_1, \vdash_7, \) and \( \vdash_e \) below. For \((\ell \sigma, r \sigma) \in T \), there are \( h \geq k \) and a substitution \( \theta \) with \((\ell \sigma) \downarrow_1 = w_0((\overline{S}^k_R \cap \leftrightarrow_{2_k^*} \cup \leftrightarrow_{2_k^*} \cap \leftrightarrow_{2_k}) = \theta \).

\[
\begin{align*}
(S, T, G_1, k) \vdash (S, T, G_1, h) & \text{ by } G_1 \vdash (u_0 \rightarrow \cdots \rightarrow u_n), \\
(S, T, G_1, h) \vdash (S, T, G_1, k') & \text{ for } w \in V_1 \text{ such that } w \text{ is } \rightarrow_1 \text{-normal, and } w \leftrightarrow_{2_k^*}, r \theta, \\
(S, T, G_1, k') \vdash (S', T', G_1', k') & \text{ for } S' = S \cup \{((\ell \theta, r \theta) \} \text{ and } T' = T \setminus \{((\ell \sigma, r \sigma) \}.
\end{align*}
\]

Let

\[
\begin{align*}
\left\{ (S, T, G_1, h) \vdash (S, T, G_1, k') \right\} & \text{ for } w \in V_1 \text{ such that } w \text{ is } \rightarrow_1 \text{-normal, and } w \leftrightarrow_{2_k^*}, r \theta, \\
(S', T', G_1', k') & \text{ for } S' = S \cup \{((\ell \theta, r \theta) \} \text{ and } T' = T \setminus \{((\ell \sigma, r \sigma) \}.
\end{align*}
\]

**Lemma 5.2.** Let \( G_0 = \langle V_0, \rightarrow_0 \rangle \) be an \( R \)-reduction graph. Then, there exists a convergent and subterm-closed \( R \)-reduction graph \( G_4 \) with \( G_0 \subseteq G_4 \).

**Example 5.3.** Let us consider to apply Lemma 5.2 on \( G_0 \) in Example 4.2. First, we take a convergent subterm-closed \( R \)-reduction graph that weakly subsumes \( \text{sub}(G_0) \). This graph is essentially the same as \( G \) in Example 3.5 containing some garbage. For simplicity, we use \( G \) in Example 3.5. As in Example 4.2, we obtain \( G_1 \) and \( k = 1 \). Let \( T = (G_0^c \cup G^c) \setminus (G_0^c \cup G^c) \), where \( G_0^c \) and \( G^c \) have the only edges \( f(c, g(c)) \rightarrow g^3(c) \) and \( c \rightarrow g(c) \), respectively.
The conversion $\vdash$ is applied twice, corresponding to two edges in $T$. The edge $c \rightarrow g(c)$ in $T$ is simply moved to $S$. For the edge $f(c,g(c)) \rightarrow g^2(c)$ in $T$, $\tau_1$ adds $f(g^2(c),g^2(c)) \rightarrow (g^2(c),g^2(c))$ to $G_1$. $\tau_2$ adds $g^3(c) \rightarrow g^4(c) \rightarrow g^5(c)$ to $G_1$ and increases $k$ to 3. $\tau_3$ adds $f(g^2(c),g^3(c)) \rightarrow g^6(c)$ to $S$. They are denoted by dotted arrows. Since $M_C(M_C(G))$ is $(g(c) \rightarrow g^2(c) \rightarrow \cdots \rightarrow g^4(c) \rightarrow g^5(c), g^6(c))$, $G_4 = (S \cup G_1 | D)$ is as in Figure 1, in which some garbage nodes are not presented.

Main Theorem  Non-$E$-overlapping and weakly shallow TRSs are confluent.

References


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