Confluence of linear rewriting and homology of algebras

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Abstract

We introduce the notion of higher dimensional linear rewriting systems for presentations of algebras, generalizing the notion of non-commutative Gröbner bases. We show how to use this notion to compute homological invariants of associative algebras, using the confluence properties of presentations of these algebras. Our method constitutes a new application of confluence in algebra.

1 Introduction

Several methods of computing homological invariants of associative algebras are based on non-commutative Gröbner bases of the ideal of relations of the algebra, [1, 3]. They consist in computing free resolutions of modules over the algebra, generated by some iterated overlaps of the leading terms of the Gröbner basis. In particular, these methods can be applied to homogeneous algebras that arise in many contexts, for instance representation theory, non-commutative geometry and mathematical physics. One of the fundamental properties in the homological description of these algebras is the Koszul property, introduced by Priddy [8] and generalized by Berger [2]. There exist methods to prove Koszulity based on Gröbner bases.

We present an extension of the categorical framework of higher-dimensional rewriting to the linear setting in order to describe Koszulity in terms of confluence. We introduce the notion of a linear polygraph encoding a presentation of an algebra by a rewriting system. Linear polygraphs do not require a monomial order, allowing more possibilities of termination orders than those associated with Gröbner bases. Therefore, we improve the known methods using Gröbner bases to prove Koszulity. Finally, the higher-dimensional rewriting allows us to refine methods to prove Koszulity using the homotopy reduction on convergent rewriting systems developed in [6].

In this note, we consider the case of algebras. A more general case is developed in [4].

2 Linear rewriting

2.1 Linear 2-polygraph

We fix a base field $\mathbb{K}$. A *linear 2-polygraph* Λ consists in a data (Σ_1, Λ_2) made of
- a set Σ_1, that we will suppose finite, say $\Sigma_1 = \{x_1, \ldots, x_k\}$,
- a vector space Λ_2 equipped with two linear maps s and t (source and target) from Λ_2 to the free algebra on Σ_1, denoted by Σ_1^ℓ, which is the free vector space on the set Σ_1^* of *monomials* in the variables $x_1, \ldots, x_k$. The length of the monomials induces a *weight grading* on Σ_1^ℓ. Elements in Σ_1^ℓ are called 1-cells and elements in Λ_2 are called 2-cells.

Any 1-cell f in Σ_1^ℓ can be written uniquely as a linear sum $f = \lambda_1 m_1 + \ldots + \lambda_p m_p$, where, for any $1 \leq i \leq p$, $\lambda_i \in \mathbb{K} \setminus \{0\}$ and m_i is a monomial 1-cell. A 2-polygraph Λ is said to be *monic* if the vector space Λ_2 has a basis Σ_2 such that any 2-cell in Σ_2 has a monomial source. For $N \geq 2$, a *monic N-homogeneous* linear 2-polygraph is a linear 2-polygraph Λ, such that any 2-cell in Σ_2 has the form $\alpha : m \Rightarrow \lambda_1 m_1 + \ldots + \lambda_p m_p$ with m and the m_i's are in weight N.

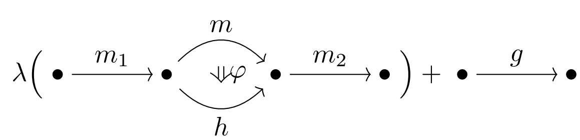
Example 2.1. Consider the algebra $\mathbf{A}$ with generators x, y, z and the relation $x^3+y^3+z^3 = xyz$ in weight 3. The monic 3-homogeneous linear 2-polygraph Λ defined by $\Sigma_1 = \{x, y, z\}$ and $\Sigma_2 = \{xyz \Rightarrow x^3 + y^3 + z^3\}$ presents the algebra $\mathbf{A}$.

In this note, Λ denotes a monic linear 2-polygraph and $\mathbf{A}$ denotes the algebra presented by Λ.

2.2 Rewriting properties of linear 2-polygraphs

The rewriting paths of a string rewriting system form a structure of a 2-category, in which the 1-cells are the strings, the 2-cells are the rewriting paths and the compositions correspond to the sequential and parallel compositions of rewriting paths, as described in [6]. In [4], we show that for linear 2-polygraphs, the 2-category induced by the rewriting paths is linearly enriched. We denote by Λ_2^ℓ the free monoid enriched in 2-vector spaces generated by Λ. Its 1-cells are the 1-cells in Σ_1^ℓ and its 2-cells are linear combinations of all possible parallel and sequential compositions of generating 2-cells in Λ_2.

The notion of a rewriting step induced by a linear 2-polygraph Λ needs to be defined with attention owing to the invertibility of 2-cells. Indeed, given a rule $\varphi : m \Rightarrow h$ in Λ, there are 2-cells $-\varphi : -m \Rightarrow -h$ and $-\varphi+(m+h) : h \Rightarrow m$ in Λ_2^ℓ. Thus we cannot have termination if we consider all 2-cells of Λ_2^ℓ as rewriting sequences. We define a rewriting step as the application of one rule on one monomial of a free linear combination of monomials. A *rewriting step* is a 2-cell in Λ_2^ℓ with the shape $\alpha = \lambda m_1 \varphi m_2 + g$:



where $\lambda \in \mathbb{K} \setminus \{0\}$, m_1, m_2 are monomial 1-cells in $\Sigma_1^\ell \setminus \{0\}$, $\varphi : m \Rightarrow h$ is a monic rule and g a 1-cell in Σ_1^ℓ such that the monomial $m_1 m m_2$ does not appear in the basis decomposition of g.

A rewriting step α from f to f' is denoted by $\alpha : f \Rightarrow^p f'$. The relation $\Rightarrow^p$ is called the *reduction relation* induced by Λ. A *rewriting sequence* of Λ is a finite or an infinite sequence $f_1 \Rightarrow^p f_2 \Rightarrow^p f_3 \Rightarrow^p \cdots \Rightarrow^p f_n \Rightarrow^p \cdots$ of rewriting steps. We say that Λ *terminates* when it has no infinite rewriting sequence. If there is a non-empty rewriting sequence from f to g, we say that f *rewrites* into g and we denote $f \Rightarrow^* g$.

We denote by Λ_2^+ (resp. Λ_2^{+f}) the set of (resp. finite) rewriting sequences of Λ, also called *positive 2-cells* of the linear 2-polygraph Λ. We denote $f \Leftrightarrow^* g$ when there exists a finite zigzag of rewriting steps between f and g.

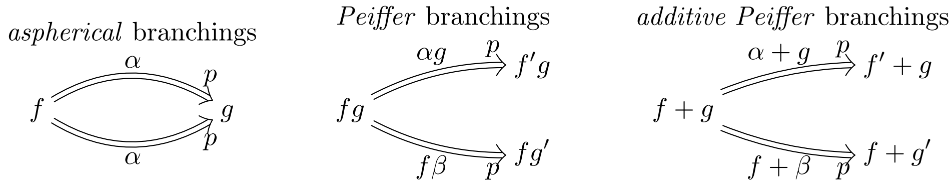
We denote by $I(\Lambda)$ the two-sided ideal of the algebra $\mathbf{A}$ generated by $\{\ m - h \mid m \Rightarrow h \in \Lambda_2^\ell\ \}$. Given 1-cells f and f' in Λ_1^ℓ, there is a 2-cell $f \Rightarrow f'$ in Λ_2^ℓ if and only if $f - f' \in I(\Lambda)$.

A 1-cell f of Λ_1^ℓ is *irreducible* when there is no rewriting step for Λ with source f. A *normal form* of f is an irreducible 1-cell g such that f rewrites into g. A 1-cell in Λ_1^ℓ is *reducible* if it is not irreducible. We denote by $\mathrm{ir}(\Lambda)$ (resp. $\mathrm{ir}_{\mathrm{m}}(\Lambda)$) the set of (resp. monomials) irreducible 1-cells for Λ. The rules being monic, the set $\mathrm{ir}(\Lambda)$ forms a vector space generated by $\mathrm{ir}_{\mathrm{m}}(\Lambda)$.

When Λ terminates, the vector space Λ_1^ℓ has the following decomposition $\Lambda_1^\ell = \mathrm{ir}(\Lambda) + I(\Lambda)$.

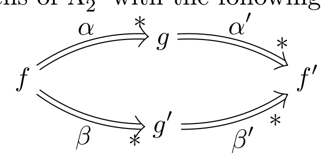
2.3 Confluence of 2-linear polygraphs

A *branching of* Λ is a pair (α, β) of 2-cells of Λ_2^+ with a common source. A branching (α, β) is *local* when α and β are rewriting steps. Local branchings belong to one of the following families:



where α and β are rewriting steps. The *overlapping* branchings are the remaining local branchings. The local branchings are compared by the strict order $\prec$ generated by $(\alpha, \beta) \prec \big(\lambda m\alpha m' + g, \lambda m\beta m' + g\big)$, for any local branching (α, β), where λ is in $\mathbb{K} \setminus \{0\}$, m, m' are monomial 1-cells in Σ_1^ℓ, g is a 1-cell in Σ_1^ℓ, no monomial in the basis decomposition appears in the basis decomposition of $ms(\alpha)m'$ and at least one of the two following conditions: 1/ either m or m' is not an identity monomial, 2/ the 1-cell g is not zero.

An overlapping local branching that is minimal for the order $\prec$ is called a *critical branching.* Note that the critical branchings have a monomial source. A branching (α, β) is *confluent* when there exists a pair (α', β') of 2-cells of Λ_2^+ with the following shape:



We say that Λ is *confluent* (resp. *locally confluent*) when all of its branchings (resp. local branchings) are confluent. We prove that a linear 2-polygraph is locally confluent if and only if all its critical branchings are confluent. A linear 2-polygraph Λ is confluent if and only if for any f in $I(\Lambda)$, $f \Rightarrow^* 0$. If moreover, the linear 2-polygraph Λ is terminating, then it is confluent if and only if we have the decomposition $\Sigma_1^\ell = \mathrm{ir}(\Lambda) \oplus I(\Lambda)$.

A linear 2-polygraph is said to be *convergent* when it is terminating and confluent. Such a polygraph is called a *convergent presentation* of the algebra $\mathbf{A}$. In this case, there is a canonical section $\mathbf{A} \to \Sigma_1^\ell$ sending f to its normal form denoted by $\widehat{f}$, so that $\widehat{f} = \widehat{g}$ holds in Σ_1^ℓ if and only if we have $f = g$ in $\mathbf{A}$. Thus, by the decomposition $\Sigma_1^\ell = \mathrm{ir}(\Lambda) \oplus I(\Lambda)$, the set of irreducible monomials $\mathrm{ir}_{\mathrm{m}}(\Lambda)$ forms a $\mathbb{K}$-linear basis of the algebra $\mathbf{A}$ via the canonical map $\mathrm{ir}(\Lambda) \longrightarrow \mathbf{A}$, called a *standard basis* of $\mathbf{A}$. In [4], we show that we recover the usual notions of Gröbner basis and of PBW basis when the rules in Λ_2 are compatible with a monomial order.

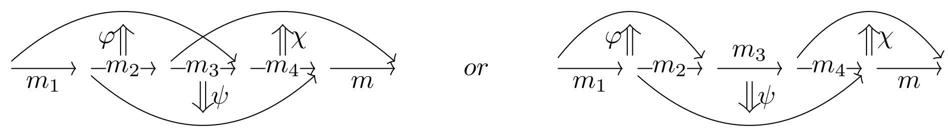
3 Polygraphic resolutions of algebras

In this section, we define polygraphic resolutions for algebras, which are extensions of presentations of an algebra in higher dimensions with a property of acyclicity. Such resolutions were introduced in [5] for monoids and categories.

Starting from a linear 2-polygraph Λ, we define the 2-spheres of Λ by the pairs (f, g) of elements in Λ_2^ℓ such that $s(f) = s(g)$ and $t(f) = t(g)$. A *linear extension* of Λ_2^ℓ is a vector space Λ_3, together with linear maps s and t from Λ_3 to Λ_2^ℓ sending an element h on a 2-sphere $(s(h), t(h))$. The data $(\Sigma_1, \Lambda_2, \Lambda_3)$ defines a *linear* 3*-polygraph.* Proceeding inductively with the notion of spheres and linear extensions, we define the notion of *linear* n*-polygraphs.*

A linear n-polygraph Λ is called *acyclic* when any k-sphere (f,g) can be filled by a $(k+1)$-cell A, that is $s(A)=f$ and $t(A)=g$, for all $1 \leq k < n$. A *polygraphic resolution* of $\mathbf{A}$ is an acyclic linear ∞-polygraph Λ, whose underlying linear 2-polygraph Λ_2 is a presentation of $\mathbf{A}$.

Consider a linear 2-polygraph Λ, which is *reduced*, that is for every 2-cell $\varphi : m \Rightarrow f$ in Λ_2, the 1-cell m is a normal form for $\Sigma_2 \setminus \{\varphi\}$ and the 1-cell f is irreducible for Λ_2. We define a *k-fold branching* by a k-tuple $(f_1,\ldots,f_k)$ of 2-cells of Λ_2^+ with the same source. As before, we can define the overlapping branchings, and an ordering relation via inclusion. We define the *critical* k-fold branchings as the minimal overlapping k-fold branchings. For instance, when $k=3$, we get two possible shapes of such critical branchings.



where the m_i's are monomials and φ,ψ,χ are generating 2-cells. The following result states that a polygraphic resolution can be built using higher critical branchings of a convergent presentation.

Theorem 3.1. *[4, 4.2.10] Any convergent linear 2-polygraph Λ extends to an acyclic linear ∞-polygraph whose k-cells, for $k \geq 3$, are indexed by the critical $(k-1)$-fold branchings.*

4 Confluence for the Koszul property

An N-homogeneous algebra $\mathbf{A}$ is called *Koszul* if there exists a free resolution of graded right $\mathbf{A}$-modules

$$0 \longleftarrow \mathbb{K} \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \ \ldots\ \longleftarrow F_{k-1} \longleftarrow F_k \longleftarrow \ \ldots$$

such that for any integer $k \geq 0$, the $\mathbf{A}$-module F_k has the form $N_k \otimes \mathbf{A}$ with all elements of N_k in weight $\ell_N(k)$, where $\ell_N(k) = lN$, if $k=2l$, and $\ell_N(k) = lN+1$, if $k = 2l+1$.

In [4], we show that polygraphic resolutions for an algebra $\mathbf{A}$ induce free resolutions of $\mathbf{A}$-modules of the base field $\mathbb{K}$. As a consequence, a necessary condition for an algebra to be Koszul can be expressed in terms of polygraphic resolutions. For this purpose, the weight grading on a linear 2-polygraph is extended on higher cells. Given a weight function $\omega : \mathbb{N} \to \mathbb{N}$, a graded polygraphic resolution Λ is *ω-concentrated* when its k-cells are concentrated in weight $\omega(k)$. The necessary condition is formulated as follows:

Theorem 4.1. *[4, 5.3.4] Let $\mathbf{A}$ be an N-homogeneous algebra. If $\mathbf{A}$ has an ℓ_N-concentrated polygraphic resolution, then $\mathbf{A}$ is Koszul.*

As a consequence of this result, the confluence property can be used to prove that an algebra is Koszul. Indeed, we have

Proposition 4.2. *[4, 5.3.6] Let $\mathbf{A}$ be an algebra presented by a quadratic convergent 2-polygraph Λ, then Λ can be extended into an ℓ_2-concentrated polygraphic resolution. In particular, such an algebra is Koszul.*

Finally, the following proposition is often be useful, in particular when using the homotopy reduction procedure on convergent rewriting systems developed in [6].

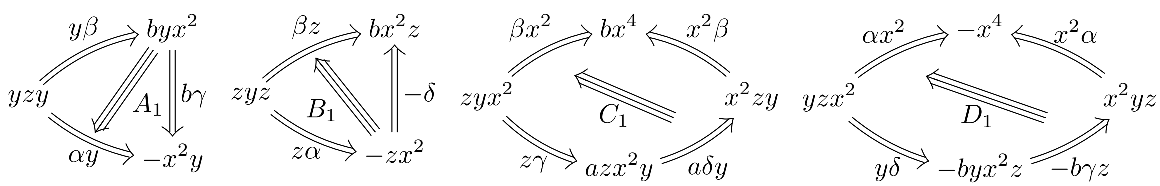
Proposition 4.3. *[4, 5.3.8] If an N-homogeneous algebra $\mathbf{A}$ is presented by an acyclic ℓ_N-concentrated 3-polygraph Λ with $\Lambda_3 = \{0\}$, then $\mathbf{A}$ is Koszul.*

We now show on some examples how our method can be applied. Given an algebra $\mathbf{A}$ presented by generators and relations, the idea is first to obtain a convergent linear 2-polygraph Λ (using usual completion procedures if necessary) presenting $\mathbf{A}$. Then we study the critical branchings of Λ to understand the polygraphic resolution and its properties.

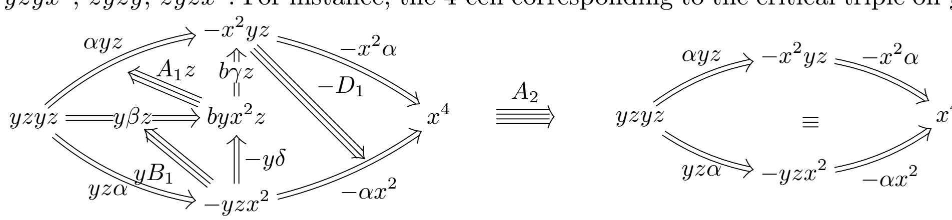
Example 4.1. The linear 2-polygraph Λ defined in the Example 2.1 is convergent, without any critical branching. It follows that $(\Sigma_1, \mathbb{K}\Sigma_2, \{0\})$ is a ℓ_3-concentrated acyclic 3-polygraph, hence by Proposition 4.3, $\mathbf{A}$ is Koszul.

Example 4.2. Consider the algebra $\mathbf{A}$ with generators x, y, z and the relations $x^2 + yz = 0$, $x^2 + azy = 0$ where $a \in \mathbb{K} \setminus \{0; 1\}$. It is a Koszul algebra, see [7, Section 4.3], but has no convergent quadratic presentation. We prove that the linear 3-polygraph Λ with $\Sigma_1 = \{x, y, z\}$,

$$\Sigma_2 = \{\ yz \overset{\alpha}{\Rightarrow} -x^2\ ,\ zy \overset{\beta}{\Rightarrow} bx^2\ ,\ yx^2 \overset{\gamma}{\Rightarrow} ax^2y\ ,\ zx^2 \overset{\delta}{\Rightarrow} -bx^2z\ \},\ \Sigma_3 = \{A_1, B_1, C_1, D_1\}$$



is acyclic and forms a presentation of the algebra $\mathbf{A}$. There are four critical triples with sources $yzyz$, $yzyx^2$, $zyzy$, $zyzx^2$. For instance, the 4-cell corresponding to the critical triple on $yzyz$ is



The 4-cell A_2 relates the 3-cell D_1 with the 3-cells A_1 and B_1. Using a homotopy reduction procedure as in [6], we can remove the cells which are not on the diagonal. Thus we obtain an ℓ_2-concentrated polygraphic resolution and by Proposition 4.3, it follows that $\mathbf{A}$ is Koszul.

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