On the Formalization of λ-Calculus Confluence and Residuals

Beniamino Accattoli

Università di Bologna, Italy

The confluence or Church-Rosser theorem is the first result in every course on the λ-calculus. Its standard proof follows Tait and Martin-Löf's technique, based on reducing confluence of β-reduction to the diamond property of its parallel closure. It is reasonably simple and yet non-trivial. Presumably because most proof assistants are built over some functional language, confluence for λ-calculus is the theorem with the highest number of formalized proofs [12, 15, 11, 16, 13, 14, 9, 4, 8, 5, 1].

On the one hand, such a formalization effort has helped to clarify the essence of the proof. On the other hand, the confluence property became a simple and yet significant benchmark for proof assistants, to test the faithfulness of the formalization to the informal, pen-and-paper reasoning on the λ-calculus. Humans can easily (but informally) work *up to equivalence or isomorphism*, while this kind of reasoning is a challenge for proof assistants. Formalizations about languages with binders, as the λ-calculus, have to face the difficulty of reasoning transparently modulo α-equivalence, *i.e.* renaming of bound variables. This issue is so relevant that in past years the *POPLmark challenge* [2]—consisting in the formalization of some theorems in the theory of λ-calculus—was proposed as a challenging benchmark for proof assistants.

In this talk I will introduce the proof assistant Abella [7, 6] and I will show a formalization of confluence that mimics *exactly* the informal reasoning. Then I will compare with formalizations in proof assistants based on different approaches to α-equivalence. Last, I will discuss some variations and refinements, including the similar proof based on finite developments (sometimes credited to Takahashi [17]), and the cube property for *residuals* [10, 3], a stronger form of diamond property for parallel reduction, having an elegant formalization based on a brilliant idea by Gérard Huet [9, 1].

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